Strategy-proof Matching with Regional Minimum Quotas*

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ABSTRACT

This paper considers the matching problem with regional quotas, in particular, regional minimum quotas. Although such quotas are relevant in many real-world settings, there is a lack of strategy-proof mechanisms that consider regional minimum quotas. We first show that without any restrictions on the region structure, finding a feasible matching that satisfies all quotas is NP-complete. Then, assuming that regions have a hierarchical structure (in this case, a tree), and maximum quotas are imposed only on individual schools, we show that checking the existence of a feasible matching can be done in a linear time in the number of regions. Furthermore, we develop strategy-proof matching mechanisms based on the Deferred Acceptance mechanism (DA), which we call Multi-Stage DA with Regional minimum Quotas (MSDA-RQ) and Round-robin Selection DA with Regional minimum Quotas (RSDA-RQ). When minimum quotas are imposed, fairness and nonwastefulness are incompatible. We prove that RSDA-RQ is fair but wasteful, while MSDA-RQ is nonwasteful but not fair. Moreover, we compare our mechanisms with artificial cap mechanisms whose individual maximum quotas are adjusted beforehand so that all regional quotas can be automatically satisfied. Our simulation reveals that our mechanisms substantially outperform artificial cap mechanisms in terms of student welfare. Furthermore, it illustrates the trade-off between our mechanisms.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multi-agent systems*; J.4 [**Social and Behavioral Sciences**]: Economics

General Terms

Algorithms, Economics, Theory

Keywords

Two-sided matching; Deferred acceptance; Minimum quotas

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1. INTRODUCTION

The theory of matching has been extensively developed for markets in which the agents (students/schools, hospitals/residents, workers/firms) have individual *maximum* quotas, i.e., the number of students assigned to a school cannot exceed a certain limit.[1] In many real-world markets, however, *minimum* quotas may also be relevant. For example, school districts may need at least a certain number of students in each school in order for the school to operate, as in college admissions in Hungary [2]. In a study on the market for Japanese medical residents [9], the Japanese government desires more doctors to be assigned to rural hospitals, and imposing minimum quotas on them is one possible approach. In the context of schools, minimum quotas may be important not only in assigning students across schools but also in assigning students to classes *within* schools. For example, in many engineering departments of Japanese universities, an undergraduate student must be assigned to a laboratory and complete his/her graduation project. Students are able to submit preferences over the labs, but each lab has certain minimum and maximum quotas that must be respected. Furthermore, diversity constraints at schools can also be considered a minimum quota problem, where school districts impose a minimum quota for each type of student at each school.

Furthermore, these minimum quotas can be imposed on a set of schools (region), rather than on individual schools. One motivating example of this model is a hospital-resident matching problem. Assume that a policy maker requires that a certain number of residents must be assigned to hospitals on an isolated island. However, she does not want to interfere with how these residents are assigned within the hospitals on the island. Also, when allocating students to labs, it is common that labs are classified into several sub-departments (courses). Achieving a good balance of the total number of allocated students for each course can be important, while the number of students in each lab can vary significantly. When a company allocates human resources to teams, it is natural to assume that the obtained matching must satisfy feasibility constraints in various levels in the organization's hierarchy, e.g., each division, department, or section has its own minimum quota. Such a feasibility constraint can be naturally represented by a regional minimum quota without restricting the space of feasible matchings.

Table 1 summarizes existing works related to regional maximum/minimum quotas. We assume that individual

[1]See [12] for a comprehensive survey of many results in the literature on this.

<table>
<tr><td></td><td></td><td colspan="3">Maximum quotas</td></tr>
<tr><td rowspan="5">Minimum quotas</td><td></td><td>Individual</td><td>Hierarchical regions</td><td>General regions</td></tr>
<tr><td>None</td><td>DA [6]</td><td>KK [9]</td><td rowspan="4">NP-complete*</td></tr>
<tr><td>Individual</td><td>FITUY [5]</td><td rowspan="2">open</td></tr>
<tr><td>Hierarchical regions</td><td>MSDA-RQ*
RSDA-RQ*</td></tr>
<tr><td>General regions</td><td colspan="2">NP-complete*</td></tr>
</table>

Table 1: DA-based mechanisms with regional quotas and $*$ indicates our contributions (KK: Kamada and Kojima, FITUY: Fragiadakis *et al.*)

maximum quotas are requisite. When considering regional quotas, we distinguish a special case where the regions have a hierarchical structure. Without regional quotas, the standard Deferred Acceptance mechanism (DA) is widely used because it is strategy-proof, fair, and nonwasteful. Also, regional maximum quotas, in which regions have a hierarchical structure, are considered in [2, 9]. Individual minimum quotas are considered in [5] and two DA-based strategy-proof mechanisms are developed. However, as shown in Table 1, a number of interesting combinations have not been explored so far. In this paper, we study the combinations with the boldface types in Table 1.

More specifically, we first analyze the complexity for finding a feasible matching that satisfies all the regional quotas, when we put no restrictions on regions. We then prove that this problem is NP-complete. Since checking the existence of a feasible matching is intractable in general, we are going to concentrate on a special case as described in Table 1, i.e., regions are hierarchical and each (non-singleton) region can impose a minimum quota only. A hierarchical structure is ubiquitous in any organization (company, university, or military). Also, it is natural to assume that multiple organizations (e.g., schools, hospitals) are geographically organized at various levels (e.g., city, county, and state). Thus, we believe that handling this special case is significant. A similar hierarchical model is used in [2, 9].

For this case, we develop two mechanisms called Multi-Stage Deferred Acceptance mechanism with Regional minimum Quotas (MSDA-RQ) and Round-robin Selection Deferred Acceptance mechanism with Regional minimum Quotas (RSDA-RQ). These mechanisms are strategy-proof. RSDA-RQ is fair, and MSDA-RQ is nonwasteful.

2. RELATED WORKS

There are a lot of literature on two-sided matching [12]. In recent years, matching problems with some constraints have been broadly studied, e.g., [2, 5, 9]. They extend the standard DA mechanism to their own constraints such as regional maximum quotas and individual minimum quotas. However, none of their mechanisms cannot handle the constraint of regional minimum quotas we consider.

Monte and Tumennasan [10] consider the problem of assigning agents to different projects, where each project needs more agents than a particular number. They develop a strategy-proof serial dictatorship mechanism available for this setting. Although their proposed mechanism is similar to our serial dictatorship mechanism described in Section 5.1, it cannot handle regional minimum quotas.

In the literature of computational science, several generalizations of the standard stable matching have also been proposed, and the complexity of checking the existence of a (generalized) stable matching has been discussed [3, 8]. Our setting is different from these existing works since we handle regional quotas, and our complexity result is on checking the existence of a *feasible* matching.

We develop mechanisms that can directly handle regional quotas. An alternative approach is to artificially modify individual quotas so that all regional quotas can be automatically satisfied when individual quotas are satisfied. Then we can apply any existing mechanisms that can handle only individual quotas. We call this an *artificial cap* mechanism. In Section 6, we show the advantage of our approach over artificial cap mechanisms.

3. MODEL

A market is a tuple $(S, C, R, p, q, \succ_S, \succ_C, \succ_{ML})$. $S = \{s_1, s_2, \ldots, s_n\}$ is a set of students, $C = \{c_1, c_2, \ldots, c_m\}$ is a set of schools, and $R = \{r_1, r_2, \ldots\}$ is a set of regions, each of which is simply set of schools $r \in 2^C \setminus \{\emptyset\}$. Let us denote $p = (p_r)_{r \in R}$ and $q = (q_r)_{r \in R}$ as the regional minimum and maximum quota vectors where $0 \leq p_r \leq q_r$ for all $r \in R$. When r consists of a single school, $p_{\{c\}}/q_{\{c\}}$ represent the minimum/maximum quotas of individual school p_c/q_c. Each student s has strict preference relation $\succ_s$ over the schools, while each school c has idiosyncratic strict priority relation $\succ_c$ over the students. The vectors of all such relations are denoted as $\succ_S = (\succ_s)_{s \in S}$ for the students and $\succ_C = (\succ_c)_{c \in C}$ for the schools. We assume that all schools are acceptable to all students and vice versa.[2]

In addition to the idiosyncratic school priorities, some of our mechanisms assume the existence of a separate *master list* (ML). ML defines a strict priority relation over the students, $\succ_{ML}$, which is used as a type of *tie-breaker* among them. ML may correspond to GPA or TOEFL scores, which induce a common ranking across all students. The concept of a master list is commonly used in matching literature [11]. W.l.o.g., we assume $s_1 \succ_{ML} s_2 \succ_{ML} \cdots \succ_{ML} s_n$.

A matching is mapping $\mu : S \cup C \to 2^{S \cup C}$ that satisfies the following conditions:[3] (i) $\mu(s) \in C$ for all $s \in S$, (ii) $\mu(c) \subseteq S$ for all $c \in C$, and (iii) for any s and c, we have $\mu(s) = c$ if and only if $s \in \mu(c)$. A matching is feasible if $\forall r, p_r \leq \sum_{c \in r} |\mu(c)| \leq q_r$ holds.

We introduce several desirable properties of matchings and mechanisms.

Definition 1. *Given matching μ, student s has justifiable envy toward s' who is assigned to c, if (i) $c \succ_s \mu(s)$ and (ii) $s \succ_c s'$. We say that matching μ is fair if no student has justifiable envy.*

In other words, student s would rather be matched to school c than her current match $\mu(s)$, and she has higher priority at c than student s'.

[2]If students are allowed to report schools as unacceptable, it would be impossible to guarantee the existence of a feasible matching that satisfies all regional minimum quotas and that is individually rational for the students, even if there exist sufficiently many students.

[3]To be more precise, $\mu(s)$ should be a set that consists of a single school, rather than the school itself. However, it is a common practice in matching literature to notate the school to which s is assigned as $\mu(s)$.

Definition 2. *Given matching μ, student s, who is assigned to c', claims an empty seat of c, if the following conditions hold: (i) $c \succ_s c'$, and (ii) matching μ', which is obtained from μ by moving s from c' to c, is feasible. To be more precise, $\mu'(s) = c$, and for any other student s', $\mu'(s') = \mu(s')$ hold. We say that matching μ is nonwasteful if no student claims an empty seat.*

In general, fairness and nonwastefulness cannot coexist when minimum quotas are imposed [4], i.e., no matching could be fair and nonwasteful. To guarantee the existence, we introduce a weaker version of nonwastefulness.

Definition 3. *Given matching μ, student s, who is assigned to c', strongly claims an empty seat of c if the following conditions hold: (i) $c \succ_s c'$, (ii) matching μ', which is obtained from μ by moving s from c' to c, is feasible, and (iii) $|\mu(c')| - |\mu(c)| \geq 2$. We say matching μ is weakly nonwasteful if no student strongly claims an empty seat.*

The intuitive meaning of this definition is as follows. Assume that a policy maker wants to equalize the number of students assigned to each school as much as possible. As a result, the claim of student s to obtain an empty seat is justified if by admitting it, the unbalance of the matching is strictly improved. Kamada and Kojima [9] adopt a weaker stability concept based on a similar intention.

We also introduce a weaker version of fairness.

Definition 4. *Given matching μ, student s has strongly justifiable envy toward s' if $\mu(s') \succ_s \mu(s)$, $s \succ_{\mu(s')} s'$, and $s \succ_{ML} s'$. We say matching μ is ML-fair if no student has strongly justifiable envy.*

We say a mechanism is (weakly) nonwasteful if it produces a (weakly) nonwasteful matching for every possible profile of the preferences and priorities. Similarly, a mechanism is fair if it produces a fair matching for every possible profile of the preferences and priorities.

Moreover, we define Pareto efficiency and strategy-proofness. For simplicity, we define $c \succeq_s c'$ as $c \succ_s c'$ or $c = c'$ for any $s \in S$ and any $c, c' \in C$. We say that matching μ is *Pareto efficient* if no feasible matching μ' satisfies $\mu'(s) \succ_s \mu(s)$ for some $s \in S$ and $\mu'(s') \succeq_{s'} \mu(s')$ for all $s' \in S$.

If student s claims an empty seat under matching μ, then s becomes better off under μ' and $\mu'(s') = \mu(s')$ for any student s'. Thus, μ cannot be Pareto efficient. In short, Pareto efficiency implies nonwastefulness.

We say a mechanism is *strategy-proof* if no student ever has any incentive to misreport her preference, no matter what the other students report.

4. COMPLEXITY FOR FINDING A FEASIBLE MATCHING

4.1 General case

Given an instance of a market, the first question we need to answer is whether a feasible matching exists or not. In the standard model with individual maximum quotas only, this question is easy, i.e., it suffices to check whether n is smaller than or equal to the sum of the maximum quotas. However, by introducing regional maximum/minimum quotas, this question becomes hard, i.e., it becomes NP-complete.

Theorem 1. *Given S, C, R, p, and q, checking whether a feasible matching exists or not is NP-complete. This holds even for the case where $\forall r \in R$, $|r| \leq 3$.*

Proof. Clearly, checking whether μ satisfies all quotas can be done in $O(|R| \cdot |C|)$. Thus, this problem is in NP.

To show that this problem is NP-hard, we reduce an arbitrary 3-SAT problem instance to a matching problem with regional maximum/minimum quotas. 3-SAT consists of a set of boolean variables X and a set of clauses L. Each clause l is a disjunction of three literals, each of which is either boolean variable $x \in X$ or its negation $\neg x$, e.g., $x_1 \vee \neg x_2 \vee x_3$.

Now, for a given 3-SAT problem instance, we create an equivalent matching problem instance as follows. For each literal (i.e., x or $\neg x$), we create a school. A pair of x and $\neg x$ forms a region, whose minimum/maximum quotas are 1. Also, for each $l \in L$, the schools in l form a region, whose minimum quota is 1. There are n students, where $n = |X|$. Hence, by assuming that if a student is assigned to a school, the literal is TRUE, otherwise, it is FALSE, there exists a one-to-one mapping between a feasible allocation and the instantiation of variables that satisfies all clauses. □

Note that a regional maximum quota can be represented as a regional minimum quota and vice versa. For example, assume that there exists region r, whose minimum quota is p_r. Then, we can replace it with complementary region $\bar{r} = C \setminus r$, whose maximum quota $q_{\bar{r}} = n - p_r$. This modified problem is equivalent to the original problem. Thus, the following theorem holds.

Theorem 2. *Given S, C, R, p, and q, checking whether a feasible matching exists or not is NP-complete. This holds even for the case where (i) there exist only maximum quotas, or (ii) there exist only minimum quotas.*

From Theorem 2, it is clear that obtaining any reasonable strategyproof mechanism is hopeless since we need to repeatedly check whether a complete feasible matching can be obtained from a given partial (i.e., not all students are matched to schools) matching.

4.2 Hierarchical case

Since checking the existence of a feasible matching is intractable in general, we concentrate on a special case where regions are hierarchical. Furthermore, we assume that each (non-singleton) region can impose a minimum quota only.

Definition 5 (Hierarchical region). *Set of regions R is hierarchical if $\forall r, r' \in R$ where $r \neq r'$, one of the following holds: (i) $r \cap r' = \emptyset$, (ii) $r \subset r'$, or (iii) $r' \subset r$.*

As noted at the end of the previous subsection, we can replace all minimum quotas by maximum ones. Since handling minimum quotas is considered difficult, we might imagine transforming all minimum quotas to maximum ones would be a good idea (so that we can use existing mechanisms such as [2, 9]). However, if we transform all individual minimum quotas to maximum ones, these quotas cannot be hierarchical. Thus, we cannot apply existing mechanisms to this transformed problem.

For each $c \in C$, $\{c\} \in R$, and p_c and q_c are provided, i.e., school c can impose its minimum/maximum quotas. For each $r \in R$, if $|r| > 1$, we assume that p_r is provided, but not q_r, i.e., each (non-singleton) region can impose a minimum quota only.

If set of regions R is hierarchical, we can construct a tree that represents R. We assume that C, which is the region that contains all schools, is included in R, which has non-binding minimum quota $p_C = n$, where n is the number of students.

Definition 6 (Tree). *Tree T_R for set of regions R is defined as follows: (i) root node C is the region that contains all schools, (ii) leaf node $\{c\}$ is a region that contains only one individual school $c \in C$, and (iii) for each node $r \in R$, where $r \neq C$, its parent node $r' \in R$ is a region that is the proper inclusion-minimal superset of r.*

Let us denote $children(r)$ as a set of child nodes of r. For a leaf node, i.e., $r = \{c\}$, $children(r)$ is $\emptyset$. It is clear that $r = \bigcup_{r' \in children(r)} r'$ holds for $|r| > 1$. We will often use the terms "node" and "region" interchangeably.

For each $r \in R$, where r is not a leaf node, i.e., $|r| > 1$, we define its maximum quota q_r as $\sum_{c \in r} q_c$. Note that q_r is not a parameter specified by the model. It is calculated from q_c. From this definition, it is clear that $q_r = \sum_{r' \in children(r)} q_{r'}$ holds for $|r| > 1$. We assume that $n \leq q_C$ holds, i.e., schools have enough capacity to accept all the students.

We introduce a crucial concept called a *reserved seat ticket*, which serves as the basis for checking the feasibility and building our mechanisms in the hierarchical model.

Definition 7 (reserved seat ticket). *We assume that each region $r \in R$ has a_r reserved seat tickets for the minimum quotas. a_r is recursively defined as follows: if $|r| = 1$, $a_r = p_r$, otherwise, $a_r = \max(0, p_r - \sum_{r' \subset r, r' \neq r} a_{r'})$. Also, A_r denotes the sum of the reserved seat tickets within region r, i.e., $A_r = \sum_{r' \subseteq r} a_{r'}$.*

The reserved seat tickets for region r represent its share of the total minimum quotas allocated specifically to r. By definition, $a_r = \max(0, p_r - \sum_{r' \in children(r)} A_{r'})$ holds. Also, if A_C does not exceed n, $n = p_C = \sum_{r \in R} a_r = A_C$ holds.

We extensively utilize reserved seat tickets to obtain a feasible matching. In our mechanisms, when each student is assigned to a school, she needs to consume exactly one reserved seat ticket at some region that contains the school. Conversely, no student is allowed to be assigned without getting a reserved seat ticket (even though the individual quota is not violated). Therefore, for region r, if all reserved seat tickets are consumed at any region $r' \subseteq r$, then the number of matches in r satisfies all minimum quotas within r. Subsequently, given a matching μ, if all reserved seat tickets are consumed at every region, it means that μ satisfies all regional minimum quotas.

In contrast to the general case, checking the existence of a feasible matching can be done in linear time by utilizing reserved seat tickets, i.e., the following theorem holds.

Theorem 3. *Given S, C, R, p, q, and T_R, checking whether a feasible matching exists or not is solved in linear time in the number of nodes in T_R.*

Proof. A procedure to solve this problem can be described as follows. (i) For each region r, calculate a_r and A_r in the depth-first order. (ii) If there exists $r \in R$ such that $A_r > q_r$ or $A_C > n$ holds at root node C, then there exists no feasible matching. (iii) Otherwise, there exists a feasible matching. It is clear that if there exists $r \in R$ such that $A_r > q_r$ holds, there exists no feasible matching. This is because A_r represents the minimum number of students to fill all minimum quotas within r. Also, if $A_C > n$, clearly, the number of students is not sufficient to satisfy all minimum quotas. Furthermore, if $A_r \leq q_r$ holds for each $r \in R$, and $A_C = n$ holds at root node C (since $p_C = n$, A_C cannot be strictly less than n), starting from the root node, we can divide n students into several groups and pass one group to one child, so that each regional quota is satisfied. By recursively doing this, we can decide the assignments of all individual schools. It is obvious that this procedure finishes in linear time in the number of nodes in T_R. □

5. MECHANISMS FOR HIERARCHICAL REGIONAL QUOTAS

Fragiadakis *et al.* [5] develop two group strategy-proof mechanisms called Multi-Stage DA (MSDA) and Extended-Seat DA (ESDA) that satisfy weaker stability requirements.

MSDA repeats multiple stages. In each stage, it assigns a subset of students using the standard DA (while the rest of students are reserved). The number of students assigned in each stage is chosen so that all unfilled individual minimum quotas can be filled by the reserved students, regardless of the outcome of the current stage. When the total number of unfilled individual minimum quotas is equal to the number of remaining students, then the standard DA only with individual maximum quotas is applied, in which the quota of each school c is defined as the number of students needed to meet c's minimum quota, e.g., if c's minimum quota is ten, and seven students have been assigned to c so far, we set c's maximum quota to three.

Basically, we can adopt a similar idea to MSDA for our setting. However, we need to solve the following two issues: (i) to find an appropriate number of students to be assigned in each stage, and (ii) to develop a mechanism that can be applied in the final stage, i.e., in the final stage, there still exists hierarchical regional minimum quotas as well as individual minimum quotas. We need a mechanism that can handle them.

Our new mechanism, Multi-Stage Deferred Acceptance mechanism with Regional minimum Quotas (MSDA-RQ), is an extension of MSDA, in which these two issues are addressed as follows. For (i), we develop a simple method that uses the number of reserved seat tickets at the root node as well as a more elaborate method described in Section 5.2. For (ii), we develop a mechanism called Serial Dictatorship mechanism with Regional minimum Quotas (SD-RQ), which can handle hierarchical regional minimum quotas.

In ESDA, a school is divided into two virtual schools, i.e., a standard school and an extended school. The maximum quota of a standard school is equal to the minimum quota of the original school. All extended schools create a single region. On this region, a regional maximum quota is imposed so that each standard school can be full, i.e., the original minimum quota can be satisfied. Then ESDA applies the flexible DA [9], which can handle regional maximum quotas. In our setting, however, we need to handle hierarchical regional minimum quotas as well. They cannot be converted to hierarchical regional maximum quotas. Thus, we need a mechanism that can handle hierarchical regional minimum quotas directly.

We develop a new mechanism called Round-robin Selection Deferred Acceptance mechanism with Regional minimum Quotas (RSDA-RQ), which generalizes the idea of ESDA as follows. In RSDA-RQ, school c is not divided. To be accepted into it, a student needs to obtain a reserved seat ticket provided either from individual school c or region r that includes c. If there exist only individual schools and one region that contains all individual schools (i.e., the same setting as [5]), obtaining a ticket from an individual school corresponds to being assigned to a standard school, and obtaining a ticket from the region corresponds to being assigned to an extended school. By this idea of reserved seat tickets, we can handle not only individual minimum quotas, but also hierarchical regional minimum quotas.

5.1 Serial dictatorship mechanism with regional minimum quotas

We first introduce SD-RQ, which we intend to use as a component of MSDA-RQ. The mechanism repeats the stages from 1 to n. In the k-th stage, student s_k, who is ranked k-th in ML, is selected. s_k is assigned to her most preferable school c that has not reached its individual maximum quota q_c, as well as there exists at least one region $r \ni c$, where a reserved seat ticket remains. s_k consumes one reserved seat ticket of r. If multiple r's exist, choose the closest one to leaf node $\{c\}$.

Let d be the depth of a given tree. SD-RQ runs in $O(n \cdot m \cdot d)$, since each student needs $O(d)$ steps to check the constraints for at most m schools.

The following theorem holds.

THEOREM 4. *SD-RQ is strategy-proof, ML-fair, and always produces a feasible and Pareto efficient matching.*

PROOF. It is clear that SD-RQ is ML-fair. It always produces a feasible matching, since at each stage, the number of remaining reserved seat tickets is equal to the number of remaining students. As a result, there exists at least one school where s_k can be assigned, and the obtained matching satisfies all minimum/maximum quotas. Also, since SD-RQ is one instance of serial dictatorship mechanisms, in which each student sequentially acts as a dictator and chooses her most favorite outcomes, it is automatically strategy-proof and Pareto efficient [1]. $\square$

5.2 Multi-stage deferred acceptance mechanism with regional minimum quotas

MSDA-RQ is an extension of MSDA [5]. At stage 1 of MSDA-RQ, we select group of students $\bar{S}^1$ and run standard DA on it, while remaining students $S^1 = S^0 \setminus \bar{S}^1$ are reserved (where $S^0 = S$). Let us define $e^1 = |\bar{S}^1| = a_C$, i.e., the number of reserved seat tickets at root node C (we discuss an alternative method to determine e^k later in this section). Also, $\bar{S}^1$ is chosen as $\{s_1, s_2, \dots, s_{e^1}\}$, i.e., a set of e^1 students who ranked at the top of ML, second, and so on. After running DA on the students in $\bar{S}^1$, we let the current matching μ^1 be final and obtain new individual quotas, i.e., $q_c^k = q_c^{k-1} - |\mu^k(c)|$, where q_c^k is the individual maximum quota of c at stage k and $q_c^0 = q_c$. Also, reserved seat tickets are consumed according to μ^k in an order closer to the leaf node. Then we repeat the process analogously for the subsequent stages.

To be more precise, MSDA-RQ starts by setting $S^0 = S$ and $q_c^1 = q_c$ for all $c \in C$, and repeats the following stages.

Stage $k \geq 1$

1. Set $\bar{S}^k$ to the set of e^k students in S^{k-1} with highest priorities according to $\succ_{ML}$, where e^k is the number of remaining reserved seat tickets at the root node.
   (a) If $e^k > 0$ then run the standard DA on the students in $\bar{S}^k$ (note that only maximum quotas are considered in the standard DA, i.e., all minimum quotas are ignored).
   (b) If $e^k = 0$ and $S^{k-1} \neq \emptyset$ (thus $S^k = S^{k-1} \neq \emptyset$), then run SD-RQ for the students in S^k. Terminate the mechanism.
   (c) If $e^k = 0$ and $S^{k-1} = \emptyset$, then terminate the mechanism.
2. Obtain partial matching μ^k from step 1.
3. Apply μ^k and obtain new individual quotas q_c^{k+1} and the number of remaining reserved seat tickets.

The last stage is typically case (b) in step 1. Another possibility is case (c) in step 1, i.e., in stage $k-1$, the number of reserved seats at the root node is equal to the number of remaining students. This fact means that all minimum quotas (except C) are already satisfied. In this case, all students are allocated by the standard DA at stage $k-1$.

MSDA-RQ runs in $O(n \cdot m \cdot d)$, since the number of offers is $n \cdot m$ in the worst case, each of which requires $O(d)$ steps to check the constraints.

EXAMPLE 1. *There are 8 students, $S = \{s_1, \dots, s_8\}$, and 4 schools, $C = \{c_1, c_2, c_3, c_4\}$. Set of regions R is given as $\{\{c_1, c_2, c_3, c_4\}, \{c_1, c_2\}, \{c_3, c_4\}, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}\}$. For each school c, minimum quota p_c is 1. The maximum quota for c_1, i.e., q_{c_1}, is 1, and for each school except c_1, its maximum quota is 4. For the other regions, the minimum quotas are $p_{\{c_1,c_2\}} = 2$ and $p_{\{c_3,c_4\}} = 4$. Notice that the minimum quota for root region $p_{\{c_1,c_2,c_3,c_4\}}$ is set to 8. Thus, by Definition 7, each of regions $\{c_1\}, \dots, \{c_4\}$ has one reserved seat ticket, region $\{c_1, c_2\}$ has no reserved seat ticket, region $\{c_3, c_4\}$ has two reserved seat tickets, and root node C has two reserved seat tickets.*

The preferences and priorities are defined as follows:

$$\succ_{s_1}, \succ_{s_2}, \succ_{s_3}, \succ_{s_4}:\ c_1\ \ c_2\ \ c_3\ \ c_4,$$
$$\succ_{s_5}, \succ_{s_6}, \succ_{s_7}, \succ_{s_8}:\ c_2\ \ c_1\ \ c_4\ \ c_3.$$

$$\succ_{c_1}, \succ_{c_2}:\ s_8\ \ s_7\ \ s_6\ \ s_5\ \ s_4\ \ s_3\ \ s_2\ \ s_1,$$
$$\succ_{c_3}, \succ_{c_4}:\ s_1\ \ s_2\ \ s_3\ \ s_4\ \ s_5\ \ s_6\ \ s_7\ \ s_8,$$
$$\succ_{ML}:\ s_1\ \ s_2\ \ s_3\ \ s_4\ \ s_5\ \ s_6\ \ s_7\ \ s_8.$$

In stage 1 of MSDA-RQ, since the root node has two reserved seat tickets, i.e., $e^1 = 2$, we temporarily remove set of students $S^1 = \{s_3, s_4, s_5, s_6, s_7, s_8\}$ according to ML. We then run the standard DA with no minimum quotas on students s_1, s_2. At the end of this stage, the following matches are determined: s_1 to c_2, and s_2 to c_1, respectively.

In stage 2, there are six students remaining, and $e^2 = 2$. Thus, we temporarily remove set of students $S^2 = \{s_5, s_6, s_7, s_8\}$. We run DA with no minimum quotas on students s_3, s_4. Then, s_3, s_4 are assigned to school c_2.

In stage 3, there are four students remaining, and $e^3 = 0$. Then we run SD-RQ on the remaining students. Since reserved seat tickets are only available at schools within $\{c_3, c_4\}$, the remaining students can only apply to a school within $\{c_3, c_4\}$. At the end, the obtained matching becomes:

$$c_1:\ s_2 \qquad\qquad c_3:\ s_8$$
$$c_2:\ s_1\ \ s_3\ \ s_4 \qquad\qquad c_4:\ s_5\ \ s_6\ \ s_7.$$

THEOREM 5. *MSDA-RQ is strategy-proof, ML-fair, and always produces a feasible and nonwasteful matching.*

PROOF. Strategy-proofness and ML-fairness follow because the standard DA and SD-RQ are strategy-proof and ML-fair, the assignment is determined within each stage, and, with the preferences of the other students fixed, no student s can affect the stage on which she participates. Nonwastefulness holds because the only time student s would be unable to get into a school with empty seats is the last stage of the mechanism where SD-RQ is applied. In this case, she is forced to go to school c, where there exists $r \ni c$ that will be filled exactly to its minimum quota p_r. Thus, s cannot be feasibly moved.

Let us show that MSDA-RQ always produces a feasible matching. Since we set e^k to the number of the remaining reserved seat tickets at the root node, regardless of μ^k, each agent can obtain a reserved seat ticket, either from C or $r \subset C$. As long as case (a) in step 1 is applied, for each student s, there exists at least one school where s can be assigned since $q_C \geq n$. Once case (c) in step 1 is applied, the number of remaining reserved seat tickets in total is equal to the number of remaining students. Thus, for each student s, there exists at least one school where s can be assigned. Thus, MSDA-RQ always returns a matching so that all students can be assigned to a school. Clearly, the matching satisfies all minimum/maximum quotas. $\square$

So far, we define e^k as the number of remaining reserved seat tickets at the root node. Actually, this is too conservative; we can assign more students at each stage. An alternative way to calculate e^k is as follows. For each region r, let us define e_r^k recursively as follows: $e_r^k := a_r^k + \min_{r' \in children(r)} e_{r'}^k$, where a_r^k is the number of remaining reserved seat tickets at region r. Then e^k is given as e_C^k. Due to space limitations, we omit the proof but we can still guarantee that MSDA-RQ always produces a feasible matching by this modification. In Section 6, we use this modified version of MSDA-RQ for evaluation.

5.3 Round-robin selection deferred acceptance mechanism with regional minimum quotas

RSDA-RQ repeats the following stages. Each stage is very similar to a stage of the standard DA, but schools (deferred) accept a student one by one based on a predefined round-robin ordering within C. W.l.o.g., we assume that the ordering is $c_1, c_2, \ldots, c_m, c_1, c_2, \ldots$.

Stage $k \geq 1$

1. If all students are deferred-accepted, then make the current deferred accepted pairs a final matching and terminate the mechanism. Otherwise, initialize the reserved seat tickets and the current deferred accepted pairs, and let each student s apply to her most preferable school c from which she has not been rejected yet.
2. If all students are deferred-accepted or rejected, return to 1. Otherwise, choose next school c according to the round-robin ordering.
3. Choose student s, who is applying to c and not deferred-accepted or rejected yet, and has the highest priority according to $\succ_c$. If there exists no such student, return to 2.
4. If the number of students deferred-accepted by c is less than q_c and $\exists r \ni c$, where a reserved seat ticket is available, then c deferred-accepts s and one reserved seat ticket is consumed (if there exist multiple regions where reserved seat tickets are available, we choose the region that is closest to the leaf node), and return to 2. Otherwise, reject all students who are applying to c and not deferred-accepted yet, and return to 2.

Similarly to MSDA-RQ, RSDA-RQ runs in $O(n \cdot m \cdot d)$.

EXAMPLE 2. *Consider the same instance of Example 1. In stage 1 of RSDA-RQ, each student applies to her most preferable school, i.e., s_1, s_2, s_3, and s_4 apply to c_1, and s_5, s_6, s_7, and s_8 apply to c_2. Then, according to the round-robin ordering, c_1 is chosen and c_1 deferred-accepts s_4, who has the highest priority according to $\succ_{c_1}$. Next, c_2 is chosen and c_2 deferred-accepts s_8, according to $\succ_{c_2}$. No student applies to c_3 or c_4. Then, c_1 is chosen, again but its maximum quota has already been reached. Thus, s_1, s_2, and s_3 are rejected. The remaining students can only apply to c_2. c_2 deferred-accepts s_7 and s_6. Then, no reserved seat ticket is available for $\{c_1, c_2\}$. Thus, s_5 is rejected from c_2.*

In stage 2, the rejected students in stage 1 apply to their second preferable schools (the accepted students apply to the same school as in stage 1). Thus, s_4 and s_5 apply to c_1, and s_1, s_2, s_3, s_6, s_7, and s_8 apply to c_2. Then, according to the round-robin ordering, c_1 is chosen and c_1 deferred-accepts s_5, who has the highest priority according to $\succ_{c_1}$. Next, according to the round-robin ordering, c_2 is chosen and deferred-accepts s_8, who has the highest priority according to $\succ_{c_2}$. No student applies to c_3 or c_4. Then, c_1 is chosen again but its maximum quota has already been reached. Thus, s_4 is rejected. The remaining students can only apply to c_2. c_2 deferred-accepts s_7 and s_6. Then, no reserved seat ticket is available for $\{c_1, c_2\}$. Thus, s_1, s_2, and s_3 are rejected from c_2.

In stage 3, the rejected students in stage 2 apply to their next preferable schools. Thus, s_1, s_2, and s_3 apply to c_3 and s_4 applies to c_2. Then, c_1 deferred-accepts s_5, c_2 deferred-accepts s_8, c_3 deferred-accepts s_1, c_2 deferred-accepts s_7, c_3 deferred-accepts s_2, c_2 deferred-accepts s_6, and c_3 deferred-accepts s_3. Then, c_2 is chosen. However, no reserved seat ticket is available for regions that include c_2. Thus, s_4 is rejected.

In stage 4, s_4 applies to c_3 but is rejected since no reserved seat ticket is available for regions that include c_4.

In stage 5, s_4 applies to c_4. Then all students have been deferred-accepted and the mechanism terminates. At the end, the obtained matching becomes:

$$c_1: s_5 \qquad c_3: s_1 \quad s_2 \quad s_3$$
$$c_2: s_6 \quad s_7 \quad s_8 \qquad c_4: s_4.$$

The following theorems hold.

THEOREM 6. *RSDA-RQ always produces a feasible matching.*

PROOF. If all students are matched, all minimum and maximum quotas have been satisfied. Now, let us assume that student s cannot be matched to any school, while she has applied to all schools. We can prove that the number of consumed reserved seat tickets of each region r at each stage is non-decreasing, i.e., if r consumes t tickets at stage k, it consumes at least t tickets at stage $k+1$. On the other hand, the number of students assigned to individual school c can decrease. However, this happens only when no reserved seat ticket is available for any region $r \ni c$. If s was rejected by school c, either c was full or no reserved seat ticket was available for any region $r \ni c$. Thus, when s has been rejected by all schools, no region has any remaining reserved seat tickets. This is because for each region r, at least one of the following cases holds: (i) there exists school $c \in r$, where s was rejected because no reserved seat ticket was available, (ii) there exists school $c \in r$, where c was full when s applied to c, but the number of students assigned to c has decreased, or (iii) each school $c \in r$ was full when s applied to c, and it remains full. It is clear that for each case, no reserved seat ticket is available for r. This contradicts that $n = \sum_{r \in R} a_r$. □

THEOREM 7. *RSDA-RQ is strategy-proof and fair.*

PROOF. Due to space limitations, we only show a proof sketch. We utilize a matching-with-contract model presented in [7]. This is a very general and abstract model that can handle various generalized many-to-one matching problems including labor-market models and ascending package auctions. We re-define RSDA-RQ in this model and show that the mechanism is equivalent to the generalized Gale-Shapley mechanism (the generalized GS).

In this model, a student and a school are matched by a contract. Hence, a (many-to-one) matching between students and schools is identified as a subset of contracts where each student s can have at most one related contract while each school c can have at most q_c related contracts. Basically, this model cannot handle regional or individual minimum quotas. However, by modifying it in the following way, we can apply it in our setting. First, we suppose that set of all schools C is the only agent on the opposite side of the students. Next, we suppose "agent" C has a particular preference over the sets of contracts, where C respects the original priorities of each school and strongly dislikes any violation of the quotas. In this way, we can incorporate minimum quotas into C's preference. Then we define a choice function of C according to this preference, which provides the most preferable subset of a given set of contracts. We show that, under the choice function of C (as well as the choice function of each student), the generalized GS is equivalent to RSDA-RQ. Also, we show that the choice function of the contracts satisfies two properties, i.e., *the substitutes* condition and *the law of aggregate demand*, which are sufficient to guarantee that the generalized GS is strategy-proof. Also, the re-defined RSDA-RQ is guaranteed to produce a *stable* matching in this model, which we call Hatfield-Milgrom (HM) stable. The fact that RSDA-RQ is fair is immediate from the HM-stability, i.e., if student s has a justifiable envy, s and C form a blocking pair. Thus, such a student does not exist. □

For incompatibility, RSDA-RQ is wasteful since it is fair. However, we can say it is *weakly nonwasteful*.

THEOREM 8. *RSDA-RQ is weakly nonwasteful.*

PROOF. The fact that RSDA-RQ is weakly nonwasteful is also immediate from the HM-stability, i.e., if student s strongly claims an empty seat, s and C form a blocking pair. Thus, such a student does not exist.

A more intuitive explanation is as follows. Assume that s, who is assigned to c', strongly claims an empty seat of school c. Then, if s has declared c as her most preferred school, then s must have been accepted to c, assuming that μ' is feasible, which is a matching obtained by moving s from c' to c. This contradicts the fact that RSDA-RQ is strategy-proof. □

6. EVALUATION

This section evaluates our newly developed mechanisms. We consider a market with $n = 512$ students and $m = 64$ schools. The individual maximum quota for each school is $q_c = 40$. Thus, for most of the cases, the individual maximum quota will not be a binding constraint. The hierarchical structure of the regions is represented as a binary tree with m nodes.[4] Also, the individual minimum quota for each school is $p_c = 0$. Then, regional minimum quotas are determined so that each region except C has (roughly) the same number of reserved seat tickets. The sum of the reserved seat tickets for all regions except C, i.e., $\sum_{r \in R \setminus \{C\}} a_r$, varies from 64 to 448. The increase in $\sum_{r \in R \setminus \{C\}} a_r$ implies that the constraint of the regional minimum quotas becomes more severe.

We generate student preferences as follows. We draw one common vector u_c of the cardinal utilities from set $[0, 1]^m$ uniformly at random. We then randomly draw private vector u_s of the cardinal utilities from the same set, again uniformly at random. Then, we construct cardinal utilities over all m schools for student s as $\alpha u_c + (1 - \alpha)u_s$, for some $\alpha \in [0, 1]$. We then convert these cardinal utilities into an ordinal preference relation for each student. The higher the value of α is, the more correlated the student preferences are. In this experiment, we set α to 0.6.[5] School priorities $\succ_c$ are drawn uniformly at random, and ML is w.l.o.g. set to $s_1 \succ_{ML} \cdots \succ_{ML} s_n$. We create 100 problem instances for each parameter setting.

We compare our mechanisms to artificial cap mechanisms, i.e., AC-DA and AC-ESDA. In AC-DA, we equally set the maximum quota of each school to 8 and apply the standard DA. In AC-ESDA, we set the minimum quota of each school to the average of total reserved seat tickets for all regions except C, i.e., $\sum_{r \in R \setminus C} a_r / m$, and apply ESDA [5], which can handle individual minimum quotas.[6] Thus, AC-DA and AC-ESDA produce feasible matchings satisfying all regional minimum quotas, but those are less flexible compared to our mechanisms.

Figure 1 shows the number of students with traditional justifiable envy. The x-axis denotes the number of the reserved seat tickets for all regions except C ($\sum_{r \in R \setminus C} a_r$), and the y-axis denotes the average ratio of students with envy. Since RSDA-RQ, AC-ESDA, and AC-DA are fair, no student has justifiable envy. MSDA-RQ performs better than SD-RQ regardless of the number of the reserved seat tickets. We can see that students are more likely to have envy as the number increases, i.e., the constraint of the minimum quotas becomes more severe.

Then, Figure 2 shows the ratio of students who claim empty seats (Def. 2). Since MSDA-RQ and SD-RQ are non-

[4]We investigate several tree structures, e.g., one with small depth or some large regions containing many schools and confirmed that the qualitative tendency is very similar.

[5]Changing the value of α, we can obtain the similar results.

[6]We can create another artificial cap mechanism based on MSDA, but this mechanism is neither fair nor nonwasteful.

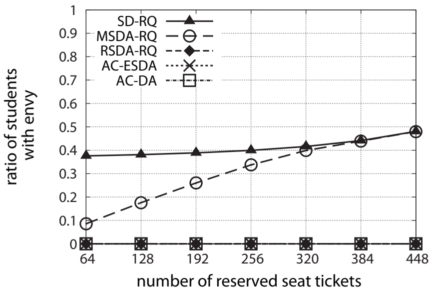

Figure 1: Ratio of students with envy

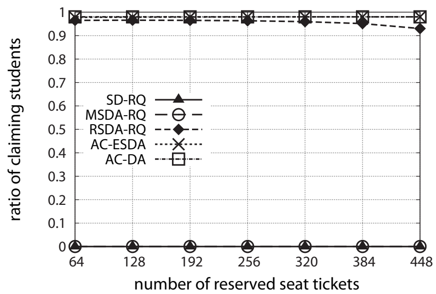

Figure 2: Ratio of claiming students

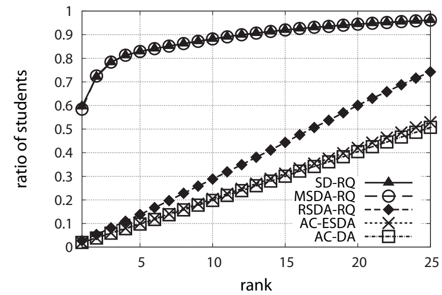

Figure 3: CDFs of student welfare

wasteful, no student claims an empty seat. In this figure, the x-axis is the same as Figure 1, and the y-axis shows the ratio of students who claim empty seats. The result reveals that AC-DA is quite wasteful; almost all students claim empty seats. RSDA-RQ is slightly better than AC-DA. Although the difference between AC-ESDA and AC-DA seems small, as described below, this small difference matters in terms of the welfare of students.

Figure 3 illustrates the student welfare by plotting the cumulative distribution functions (CDFs) of the average number of students matched with their k^{th} or higher ranked school under each mechanism. Thus, if the CDF of one mechanism first-order stochastically dominates another, then a strong argument can be made for the use of the stochastically dominant mechanism. MSDA-RQ and SD-RQ are clear winners. For example, nearly 70% of students are assigned to their first or second choice. RSDA-RQ is worse than MSDA-RQ and SD-RQ, but much better than AC-ESDA and AC-DA. The decrease of welfare seems to be the price to achieve fairness. Setting artificial caps significantly lowers the welfare of students, since we lose too much flexibility.

The experimental results clearly show that there is a trade-off between MSDA-RQ and RSDA-RQ, as mentioned in the previous section. If we consider fairness among students essential, we should use RSDA-RQ; if our primary concern is their welfare, we should use MSDA-RQ.

Comparing MSDA-RQ and SD-RQ, these mechanisms are equally good in terms of the welfare of students. However, SD-RQ completely ignores the priorities of the schools, which causes a high ratio of students with envy. Our experiments suggest that MSDA-RQ is the better choice if we care about the fairness of mechanisms to a certain extent.

7. CONCLUSIONS

This paper analyzed the complexity of finding a feasible matching for a given matching problem with regional quotas. We showed that, when we put no restrictions on the structure of regions, checking the existence of a feasible matching that satisfies all quotas is NP-complete. Then, assuming that regions have a hierarchical structure, we developed strategy-proof matching mechanisms for handling regional minimum quotas called RSDA-RQ and MSDA-RQ. We proved that RSDA-RQ is fair but wasteful, while MSDA-RQ is nonwasteful but not fair. We then confirmed the advantages of these mechanisms compared to artificial cap mechanisms via simulations.

In the future, we would like to design a mechanism for a situation where both regional minimum and maximum quotas constrain a feasible matching, i.e., the cells marked "open" in Table 1. Also, we would like to examine more theoretical properties of MSDA-RQ and RSDA-RQ. For example, we would like to show that each mechanism produces a student optimal matching within a set of matchings that satisfies a weaker stability condition.

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