How the Number of Strategies Impacts the Likelihood of Equilibria in Random Graphical Games

Anisse Ismaili Evripidis Bampis Nicolas Maudet Patrice Perny

Sorbonne Universités, UPMC Univ Paris 06, UMR 7606, LIP6, F-75005, Paris, France
{firstname.lastname}@lip6.fr

ABSTRACT

This paper studies the likelihood of the existence of a pure Nash equilibrium (PNE) in random payoff graphical games. Here, players are represented by vertices, they choose a strategy in finite discrete sets of strategies, and the scope of a player's utility function is only local. In this setting, the probability of existence of a PNE has been deeply studied for various graphical structures when the number of players tends to infinity, but only in the two strategies-per-player case: this paper extends these studies to an arbitrary number of strategies-per-player. We prove theoretically how more strategies-per-player makes the distribution of the number of equilibria get closer to a Poisson distribution. We apply these results to various graph structures and conclude with numerical experiments.

Categories and Subject Descriptors

I.2.11 [**Distributed AI**]: Multiagent Systems

General Terms

Theory, Experimentation

Keywords

Game Theory

1. INTRODUCTION

The Internet and networks have given rise to new economic contexts where a large number of self interested players interact (e.g. crowd markets, routing networks): there are growing interests for modelling and predicting what situations emerge when numerous rational behaviors interact [15, 8]. The Nash Equilibrium (NE) is a fundamental stability notion: no player has an individual incentive to deviate from his current randomized strategy, and an NE always exists. However, it is unclear why a player would adopt a randomized strategy [1]. The Pure Nash Equilibria (PNE) consider only deterministic strategies, but may exist or not.

[1]When a player chooses a mixed strategy as a best response, all strategies chosen with a non-zero probability are already necessarily a pure strategy best response for this player [16].

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In many cases, the utility functions of players do not depend on all other players, but only on a subset. Graphical games (GG) provide representations of multiplayer games involving large populations of players when influences among them are local [13, 12]. In a GG, each vertex is a player, and a player's utility function depends only on the strategies of his neighbors and his own strategy. The computation of a PNE in classes of GGs with bounded treewidth has been deeply studied [9, 2, 5, 18]. It has been proved [11] that a problem-class of PNE-GG is polynomially tractable iff this problem-class has a bounded treewidth. Otherwise PNE-GG is NP-Complete.

The existence of a PNE can be investigated by using probability measures with random payoffs and for a fixed (or random) graph. The distribution of the number of PNE in general games with random payoffs has already been studied in the following settings in n players matrix-games (i.e.: complete support graphs) [7, 17]. For short, in matrix games the probability of PNE tends to $1 - 1/e$ when the strategy sets of at least two players tend to infinite sizes. For massive networks of players, these studies are more interesting when the number of players n tends to infinity: The most recent results about GGs with various graph structures, when n tends to infinity and 2 strategies-per-player [6, 3, 10] are summarized in the Section 3. Our work extends these asymptotic studies (when the number of players tends to infinity) to the $\sigma > 2$ strategies-per-player case, with various graphs.

2. PRELIMINARIES

Given a set V, let the *power set* $\mathcal{P}(V)$ be $\{W|W \subseteq V\}$, and let $\mathcal{P}_k(V)$ denote $\{W|W \subseteq V, k = |W|\}$. In an *unoriented graph* $G = (V, E)$, the *set of vertices* is V, the *set of edges* is $E \subseteq \mathcal{P}_2(V)$, and the cardinality of $|V|$ is denoted by n. Let the *neighborhood of a vertex* v be $\mathcal{N}(v) = \{w \in V|\{w, v\} \in E\}$ and let $\mathcal{M}(v)$ denote $\mathcal{N}(v) \cup \{v\}$. More generally, for all subset of the vertices $W \subseteq V$, let the *neighborhood of* W be $\mathcal{N}(W) = (\cup_{w \in W} \mathcal{N}(w)) \backslash W$, and let $\mathcal{M}(W)$ denote $\mathcal{N}(W) \cup W$. The *degree of a subset of vertices* $W \subseteq V$ is $\delta(W) = |\mathcal{N}(W)|$. Given a subset of vertices $W \subseteq V$, the *subgraph* $G(W)$ is the restriction of G to the vertices W and the edges contained in W.

Definition 1. A graphical game (GG) is formally defined by a tuple $\Gamma = (G = (V, E), \mathcal{S} = \{S_v\}_{v \in V}, (u_v)_{v \in V})$. In the graph $G = (V, E)$ (also called support graph), each vertex represents a distinct player. A player v in V chooses a strategy s_v in his finite discrete set of strategies S_v. The utility function u_v of each player, is local. In other words,

$u_v : \prod_{w \in \mathcal{M}(v)} S_w \to \mathbb{R}$ depends only on the strategies of the players in the neighborhood of player v, and on his own strategy.

For every subset $W \subseteq V$, let us denote s_W the elements of the cartesian product $S_W = \prod_{w \in W} S_w$. We will name *local strategy profiles*, the elements $s_{\mathcal{M}(v)}$ of $S_{\mathcal{M}(v)}$ and *global strategy profiles*, the elements s_V (or s, for short) of the cartesian product S_V (or S, for short). Given a global strategy profile $s \in S$, we will use the notation s_W to denote the projection of s to S_W. We fix $0 \in S$ to be one particular global strategy profile. Each payoff $u_v(s_{\mathcal{M}(v)})$ can be defined arbitrarily in $\mathbb{R}$, for each player v, and each local strategy profile $s_{\mathcal{M}(v)}$.

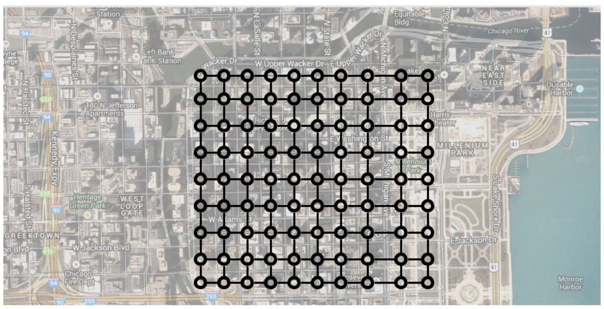

Figure 1: The Chicago-Game

Example 1. In the Chicago-Game (Figure 1), each player is the owner of one block of land, and must choose what to build among those strategies: a garden, a residential complex, a factory, a school, a museum, or a shopping mall. The payoff function of a building is local: For instance, between two factories, one would prefer to build a garden rather than a residential complex, but between some gardens and a school, one would prefer to build a museum. Will the owners agree on a building plan?

A player, given his neighbors' strategies, is considered individually stable, when his strategy is already maximizing what he can individually get for himself.

Definition 2. (Best response function, BRF). Player v's best response function $\mathrm{BR}_v : S_{\mathcal{N}(v)} \to S_v$ associates to each strategy profile $s_{\mathcal{N}(v)}$ the best response strategies:

$$\mathrm{BR}_v(s_{\mathcal{N}(v)}) = \mathrm{argmax}_{s_v \in S_v} \{u_v(s_{\mathcal{N}(v)}, s_v)\}$$

Definition 3. (Pure Nash equilibrium, PNE). A pure Nash equilibrium is a global strategy profile s such that all players are playing a best response:

$$s_v \in \mathrm{BR}_v(s_{\mathcal{N}(v)}), \text{ for all player } v.$$

In this setting, to study if a pure Nash equilibrium is likely to exist, we can use a probability measure.

Definition 4. (Random payoff graphical game, RPGG). We fix a support graph $G = (V, E)$ and a family of discrete strategy sets $\mathcal{S} = \{S_v\}_{v \in V}$ for the players. Given G and $\mathcal{S}$, a Random Payoff Graphical Game is a GG in which the payoffs $u_v(s_{\mathcal{M}(v)})$ are drawn uniformly and independently in the continuum $[0, 1]$, for all $v \in V$ and for all $s_{\mathcal{M}(v)} \in S_{\mathcal{M}(v)}$. It defines a probability measure $\mathbb{P}_{G,\mathcal{S}}$ over the GGs of support graph G and strategy sets $\mathcal{S} = \{S_v\}_{v \in V}$.

Drawing the payoffs uniformly and independently is motivated by the *simplicity* of this model for payoffs.

Given $\mathbb{P}_{G,\mathcal{S}}$, let us now consider for each global strategy profile s and each player v, a random variable $X_{s,v} \in \{0, 1\}$ which value is 1 iff $s_v \in \mathrm{BR}_v(s_{\mathcal{N}(v)})$; it indicates whether v plays a best response in the global strategy profile s. Let us consider for each $s \in S$, a random variable $Y_s \in \{0, 1\}$ which value is 1 iff the global strategy profile s is a PNE. We have $Y_s = \min_{v \in V} X_{s,v}$. We will abuse notation $X_{s,v}$ to denote the event $X_{s,v} = 1$. Similarly, $Y_s = \cap_{v \in V} X_{s,v}$ will denote $Y_s = 1$. The general purpose of this paper is to study the distribution $Z = \sum_{s \in S} Y_s$ describing the number of PNE with respect to $\mathbb{P}_{G,\mathcal{S}}$, given different graphs G and strategy sets $\mathcal{S} = \{S_v\}_{v \in V}$.

Remark 1. *The fact that a global strategy profile s is a PNE, depends only on the BRFs. One must not care about the precise values of the utilities.*

Remark 2. *Let us denote σ_v the finite cardinality number $|S_v|$. Given $s_{\mathcal{N}(v)}$, since the utilities $\{u_v(s_{\mathcal{N}(v)}, s_v)\}_{s_v \in S_v}$ are drawn uniformly in a continuum $[\alpha, \beta]$ (with $\alpha < \beta$) and independently, and since σ_v is finite, then the best responses are always singletons*[2]*, and we have $\mathbb{P}_{G,\mathcal{S}}(X_{s,v}) = \sigma_v^{-1}$.*

Remark 3. *Given a support graph G and strategy sets $\mathcal{S} = \{S_v\}_{v \in V}$, considering the measure $\mathbb{P}_{G,\mathcal{S}}$, the distributions of $X_{s,v}$, Y_s and Z depend only on G and $\mathcal{S}$.*

Remark 4. *The semantics of the strategies in the finite strategy sets do not matter. Let us assume that all players have the same number $\sigma \geq 2$ of strategies. The distribution of the random variables $X_{s,v}$, Y_s and Z with respect to $\mathbb{P}_{G,\mathcal{S}}$ are the same as distributions with respect to a probability measure denoted by $\mathbb{P}_{G,\sigma}$.*

This paper studies the probability of existence of a PNE $\mathbb{P}_{G,\sigma}(Z > 0)$, given support graphs G and numbers σ of strategies-per-player. Massively multiplayer games have motivated deep researches towards asymptotic studies over the number of players. In graph sequences $(G_n)_{n \in \mathbb{N}}$ where the graphs have a particular structure (e.g. paths, trees, grids, .., random) and $n = |V|$ denotes the number of players, it has been shown how the structure of the graph influences the limit of $\mathbb{P}_{G_n,2}(Z > 0)$ as n tends to infinity [6, 3]. The next section summarizes those results. Recent *experimental* evidences indicate that the number of strategies-per-player σ can strongly impact the convergence of $\mathbb{P}_{G_n,\sigma}(Z > 0)$ as n tends to infinity [10]. We will give *theoretical* explanations for the impact of the number of strategies-per-player.

3. PREVIOUS RESULTS

We summarize here the most important results of the literature [6, 3] concerning the probability of existence of a PNE in random payoff graphical games. While Theorem 1 gives a condition that makes PNE unlikely as $n \to +\infty$, Theorem 2 gives a condition which makes the number of PNE get close to a Poisson(1) distribution, implying $\mathbb{P}_{G_n,2}(Z > 0) \to 1 - 1/e$ as $n \to +\infty$.

[2]Given a countable set of random variables $\{U_i\}_{i \in \mathbb{N}}$ drawn uniformly and independently in a continuum $[\alpha, \beta]$ (with $\alpha < \beta$), the probability that at least two different variables U_i and U_j are equal, equals 0.

3.1 Players with 2 strategies: a condition for PNE impossibility

The following theorem is based on small witnesses of equilibrium impossibility: if two interracting players can never agree, then there is no chance to have a PNE. A number m of such two-players games makes the bound in Theorem 1.

Definition 5. (d-Bounded Edge.) In a graph $G = (V, E)$, a d-Bounded Edge is an edge $e = \{a, b\}$ such that both a and b have degrees bounded by d.

THEOREM 1. *[3, 4] (Likelihood of impossibility witnesses.) Let G be a support graph with at least m d-bounded edges without any common vertices. We have:*

$$\begin{aligned}\mathbb{P}_{G,2}(Z > 0) &\leq \left(1 - \left(\frac{1}{4}\right)^{2^d}\right)^m \\ &\leq \exp\left(-m\,(1/4)^{2^d}\right)\end{aligned}$$

In consequence, in numerous structures of support graphs holding a number $m = \Omega(n)$ of d-bounded edges (for instance: paths, rings, bounded degree trees, ...), with 2 strategies-per-player, the probability of having a PNE tends to 0, as the number of players increases (which seems close from empirical evidence [6, 10]). The present writing of Theorem 1 is different from the original one [3] because we discovered a small flaw in the proof of the original theorem, and Daskalakis et al. answered by a more carefull analysis [4] and this new version of Theorem 1. Following this, a natural way to approach our problem would be to generalize Theorem 1 to the $\sigma \geq 2$ strategies-per-player case. However, we will take a different direction, due to the fact that in our context ($\sigma > 2$), the convergence rate to 0 of such a bound is so slow that it has almost no consequence on Z.

3.2 The influence of the number of strategies

Some experimental evidences indicate that the number of strategies-per-player σ strongly impacts the convergence of $\mathbb{P}_{G_n,\sigma}(Z > 0)$ [10] (see Figure 2). It has been observed that in square-grids, the probability $\mathbb{P}_{G_n,2}(Z > 0)$ (e.g. $\sigma = 2$ strategies-per-player, which would fulfill the assumptions of Theorem 1) decreases as the number of players n grows, **whereas** $\mathbb{P}_{G_n,4}(Z > 0)$ (e.g. $\sigma = 4$ strategies-per-player) seems to behave more like a Poisson(1) distribution [3]. Therefore, *it is not satisfying to study only the case where the number of strategies-per-player is* $\sigma = 2$. We reproduced the experiments, with 5000 random payoff graphical games for each n and each σ, and obtained the Figure 2. We theoretically explain the influence of the number of strategies-per-player in the next sections.

3.3 Players with 2 strategies: number of PNE and Poisson

The following Theorem 2 gives a connectivity condition (named "expansion") which makes the number of PNE get close to a Poisson(1) distribution.

Definition 6. (Total Variation Distance.) The total variation distance [4] between two discrete probability distributions Z and T over $\mathbb{N}$ is defined by:

$$d(Z, T) = (1 - 1/e)^{-1}(1/2) \sum_{k \geq 0} |\mathbb{P}(Z = k) - \mathbb{P}(T = k)|$$

Definition 7. (Poisson(1) Distribution.) A discrete distribution T over $\mathbb{N}$ is a Poisson(1) distribution iff:

$$\forall k \in \mathbb{N}, \quad \mathbb{P}(T = k) = \frac{1}{e}\frac{1}{k!}$$

REMARK 5. *Let us consider a random variable Z over $\mathbb{N}$, a Poisson(1) distribution T, and a bound $B \in \mathbb{R}_{\geq 0}$. If the total variation distance $d(Z, T)$ is upper-bounded by B, then $\mathbb{P}(Z > 0)$ is in $\left[(1 - \frac{1}{e})(1 - B), (1 - \frac{1}{e})(1 + B)\right]$.*

Definition 8. ((α, β)-Expansion.) The expansion of a subset $W \subset V$ is $\mathcal{E}(W) = \{v \in V : \exists w \in W \text{ with } \{w, v\} \in E\}$. A graph $G = (V, E)$ is said to have (α, β)-expansion iff, whatever the subset $W \subset V, |W| \leq \lceil \beta |V| \rceil$, it expands into α times more vertices, that is: $|\mathcal{E}(W)| \geq \min\{\lceil \alpha |W| \rceil, |V|\}$.

THEOREM 2. *[3] (Convergence to Poisson(1) under sufficient expansion.) Let G be a support graph with n vertices, and an $\varepsilon > 0$. If G has $(\alpha, 1/\alpha)$-expansion with $\alpha = (1 + \varepsilon)\log_2(n)$ then:*

$$d(Z, T) \leq 2^{-n/2} + (\exp(n^{-\varepsilon}) - 1)$$

where Z is the distribution of the number of PNE under $\mathbb{P}_{G,\sigma}$, and T is a Poisson(1) distribution.

There are also results about random graphs drawn from the Erdös-Renyi model $\mathcal{G}(n, p)$ (where n is the number of vertices and $p \in [0, 1]$ is the probability for an unoriented edge to exist), it is shown that when there is an $\varepsilon > 0$ such that $c/n^2 < p < (1 - \varepsilon)\ln(n)/(2n)$ (medium connectivity) the probability $\mathbb{P}_{G_n,2}(Z > 0)$ converges to 0, and when $p > 2(1 + \varepsilon)\ln(n)/n$ the distribution Z of the number of PNE converges to a Poisson(1) distribution, as $n \to +\infty$.

3.4 Players with same number σ of strategies: the particular case of bipartite graphs

We summarize now results concerning specifically (augmented) bipartite support graphs [6], with respect to $\mathbb{P}_{G,\sigma}$ (where there are σ strategies-per-player).

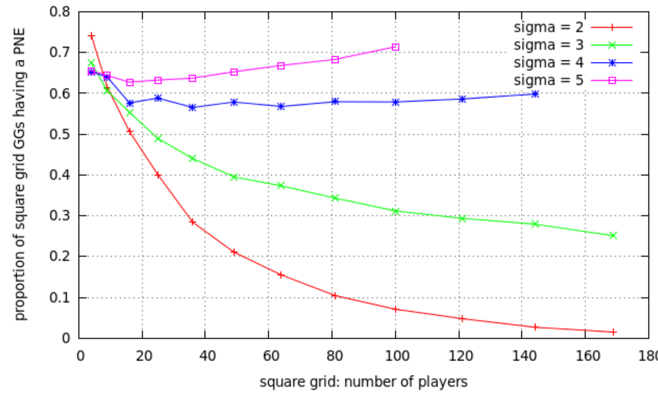

Figure 2: Proportion of square-grids having a PNE, as the number of players grows, with different numbers of strategies-per-player σ.

[3]Recall that $1 - 1/e \simeq 0.6321$

[4]Although this definition slightly differs from the previous literature [3], by a constant $(1 - 1/e)^{-1}$, a tighter use of the Lemma 2 [1] allows us to add this constant without changing the writing of the previous literature and what will follow.

THEOREM 3. *[6] Let $G_n = K(V_n^1, V_n^2)$ be a sequence of complete bipartite graphs with $|V_n^1 \cup V_n^2| = n$ vertices. If $|V_n^1|$ and $|V_n^2|$ are unbounded as $n \to +\infty$, then:*

$$\mathbb{P}_{G_n,\sigma}(Z > 0) \to 1 - 1/e, \text{ as } n \to +\infty.$$

Definition 9. A graph $G = (V^1 \cup V^2, E)$ is an augmented bipartite graph $AK(V^1, V^2, E)$ if no pair of nodes in V^2 are connected by an edge, each node in V^2 is connected to all nodes in V^1, and nodes in V^1 are arbitrarily connected.

THEOREM 4. *[6] Let $G_n = AK_n(V_n^1, V_n^2, E_n)$ be a sequence of augmented bipartite graphs with $|V_n^1| = m$ and $|V_n^1 \cup V_n^2| = n$. If $n/3 - m$ tends to $+\infty$ as $n \to +\infty$, then:*

$$\mathbb{P}_{G_n,\sigma}(Z > 0) \to 1 - (1 - (1/\sigma)^m)^{\sigma^m}, \text{ as } n \to +\infty.$$

4. STRATEGIES, CONNECTIVITY AND POISSON DISTRIBUTION ON PNE

In order to generalize Theorem 2 to the σ strategies-per-player case, we will need the technical Lemma 1 to give us a refined upper-bound on the distance between the number of PNE and a Poisson(1) distribution. This technical lemma will let us demonstrate two contributions:

- If the support-graph is connected, then more strategies-per-player can only make the number of PNE get closer to a Poisson(1) distribution.
- Under some (σ, q, r)-connectivity property which is less demanding when there are more strategies-per-player, the number of PNE converges to a Poisson(1) distribution when the number of players n tends to infinity.

4.1 A technical lemma

We first write a *technical* (counterintuitive) lemma giving an upper bound on the distance between the distribution of the number of PNE (given $\mathbb{P}_{G,\sigma}$) and a Poisson(1) distribution. The proof of Lemma 1 is in Appendix A.

Definition 10. Let $\mathcal{Q}(V) \subset \mathcal{P}(V)$ denote:

$$\mathcal{Q}(V) = \left\{ W \subset V : \begin{array}{c} W \neq \emptyset \text{ , } \mathcal{M}(W) \neq V \text{ ,} \\ \forall v \in V,\ \mathcal{N}(v) \subseteq \mathcal{M}(W) \Rightarrow v \in W \end{array} \right\}$$

LEMMA 1. *(Strong version.) Let G be a graph and let σ be the number of strategies-per-player. Let T denote a Poisson(1) distribution. Recall that for every subset of vertices $W \subseteq V$, the degree $\delta(W)$ denotes $|\mathcal{N}(W)|$. We have:*

$$d(Z,T) \leq \sum_{v \in V} \sigma^{-\delta(v)} + \sum_{W \in \mathcal{Q}(V)} \sigma^{-\delta(W)}$$

For simplicity, a *weak version* of this lemma can be obtained by: instead of summing over $W \in \mathcal{Q}(V)$, summing over the superset $\{W \subset V : W \neq \emptyset \text{ , } \mathcal{M}(W) \neq V\}$. While the weak version of Lemma 1 will let us demonstrate theoretical results in this section, the strong version will enable us to provide refined bounds for some graphical structures.

4.2 Connectivity and Poisson on PNE

Recall that $\mathcal{N}(W) = \cup_{v \in W} \mathcal{N}(v) \backslash W$ and $\delta(W) = |\mathcal{N}(W)|$. The conditions of the following Definition 11 state that *every* non-empty subset of vertices W has at least Δ other neighbors. Higher is Δ, more the graph is connected.

Definition 11. (Δ-neighborhood property.) We say that a graph $G = (V, E)$ has the Δ*-neighborhood property* if for *every* non-empty subset $W \subset V$:

$$|W| < n - \Delta \Rightarrow \delta(W) \geq \Delta$$

$$|W| \geq n - \Delta \Rightarrow W \cup \mathcal{N}(W) = V$$

REMARK 6. *The 1-neighborhood property and connectivity are equivalent. Indeed, if a graph $G = (V, E)$ is connected, then all non-empty subsets $W \subset V$ are connected to $V \setminus W$ and therefore have at least one neighbor. If a graph $G = (V, E)$ has the 1-neighborhood property, it is always possible from any vertex v to make a connected component $W \supset \{v\}$ grow iteratively to any other vertex w.*

REMARK 7. *Given a graph G, by Menger's Theorem [14], G has the Δ-neighborhood property iff G is Δ-vertex-connected (ie: between every pair of vertices, there are at least Δ vertex-disjoint paths).*

The logarithm with basis σ is denoted by $\log_\sigma(x)$.

THEOREM 5. *(Δ-convergence.) Let G be a graph and let σ be the number of strategies-per-player. Let T be a Poisson(1) distribution. If G has the Δ-neighborhood property, then:*

$$d(Z,T) \leq n\sigma^{-\Delta} + 2^n \sigma^{-\Delta}$$

Therefore, if G is connected, then it has the 1-neighborhood property, and then more strategies-per-player can only make the number of PNE Z get closer to a Poisson(1) distribution. Moreover, considering a sequence of support graphs $(G_n)_{n \in \mathbb{N}}$ such that G_n has the Δ_n-neighborhood property and denoting the number of PNE with respect to $\mathbb{P}_{G_n,\sigma}$ by Z_n, if $\Delta_n \geq \log_\sigma(2)(1+\varepsilon)n$ for some real constant $\varepsilon > 0$, then:

$$\begin{array}{rcl} d(Z_n, T) & \leq & n2^{-n} + 2^{-\varepsilon n} \\ d(Z_n, T) & \to & 0 \text{ as } n \to +\infty \end{array}$$

PROOF. The bound $n\sigma^{-\Delta} + 2^n\sigma^{-\Delta}$ is a simple consequence of the weak version of Lemma 1, combined with the Δ-neighborhood property. Remark 6 and some calculus give the other consequences. □

4.3 (σ, q, r)-connectivity and Poisson on PNE

The following Theorem 6 refines the Theorem 5 by formulating a condition which makes the number of PNE tend to Poisson(1) *as the number of players tends to infinity.*

Definition 12. ((σ, q, r)-connectivity.) Let G be a graph and let σ be the number of strategies-per-player. The graph G is (σ, q, r)-connected iff:

$$\forall W \subset V,\ \delta(W) \geq \min\{\alpha|W|, |V|\}$$

with $\alpha = (1+q)\log_\sigma(n) + r$ and $q \geq -1$ and $r \geq 0$.

THEOREM 6. *Let G be a graph and let σ be the number of strategies-per-player. Let T be a Poisson(1) distribution. If G is (σ, q, r)-connected, then:*

$$d(Z,T) \leq n^{-q}\sigma^{-r} + \left(\exp\left(n^{-q}\sigma^{-r}\right) - 1\right)$$

Therefore, considering a sequence of support graphs $(G_n)_{n \in \mathbb{N}}$, with $\alpha_n = (1+q)\log_\sigma(n) + r_n$, and denoting the number of PNE with respect to $\mathbb{P}_{G_n,\sigma}$ by Z_n, we have:

1. If $q > \varepsilon$ for some real constant $\varepsilon > 0$ and $r_n \geq 0$, Then $d(Z_n, T) \to 0$ as $n \to +\infty$

2. If $r_n \to +\infty$ as $n \to +\infty$ and $q \geq -1$, Then $d(Z_n, T) \to 0$ as $n \to +\infty$

PROOF. From the weak version of the Lemma 1, we have:

$$d(Z,T) \leq \sum_{v \in V} \sigma^{-\delta(v)} + \sum_{k=1}^{n-1} \sum_{W \in \mathcal{P}_k(V)}^{\mathcal{M}(W) \neq V} \sigma^{-\delta(W)}$$

First, since G is (σ, q, r)-connected, we have:

$$\sum_{v \in V} \sigma^{-\delta(v)} \leq \sum_{v \in V} \sigma^{-((1+q)\log_\sigma(n)+r)} = n^{-q}\sigma^{-r}$$

Second, we also have:

$$\begin{aligned} \sum_{W \in \mathcal{P}_k(V)}^{\mathcal{M}(W) \neq V} \sigma^{-\delta(W)} &\leq \binom{n}{k} \sigma^{-((1+q)\log_\sigma(n)+r)k} \\ &\leq \frac{n^k}{k!} n^{-(1+q)k} \sigma^{-rk} = \frac{\left(n^{-q}\sigma^{-r}\right)^k}{k!} \end{aligned}$$

By identifying the power series of $\exp\left(n^{-q}\sigma^{-r}\right)$, we have:

$$\sum_{k=1}^{n-1} \frac{\left(n^{-q}\sigma^{-r}\right)^k}{k!} \leq \exp\left(n^{-q}\sigma^{-r}\right) - 1$$

The other consequences follow directly. □

The conditions of the Theorem 6 formulate an expansion of every subset of vertices. As σ grows, these conditions are easier (since $\log_{\sigma+1}(x) \leq \log_\sigma(x)$) and the bound on $d(Z,T)$ is smaller, speeding up the convergence when n tends to infinity. The (σ, q, r)-connectivity is a too strong condition for graphical structures like rings, paths, trees, and grids.

5. DETERMINISTIC GRAPHS

Rings, paths, binary trees, Halin graphs (obtained from a binary tree by adding one big cycle on its leaves), square-grids and many other deterministic support graphs do not fulfill the conditions of the Theorem 6. However, they still seem to exhibit a Poisson(1) behavior for the PNE number, when the number of strategies-per-player σ is greater than two. In fact, for these structures, most subsets $W \subset V$ are highly connected and only a few subsets $W \subset V$ have a small number of neighbors: it calls for a refined use of Lemma 1.

We will do experiments for various numbers n of players and σ of strategies-per-player: We compute the intervals given by Lemma 1[5] and an estimation of $\mathbb{P}_{G,\sigma}(Z > 0)$. These intervals, denoted by [inf, sup], are evaluated for rings and paths by the combinatoric bounds of Theorems 7 and 8 (see Appendix B), unknown for binary trees and Halins, and for square grids: exactly computed, from Lemma 1. Bounds below 0 or above 1 are marked by '-'. Each $\mathbb{P}_{G,\sigma}(Z > 0)$ value (denoted by 'P') is estimated by 200 random instances that are solved by the junction tree algorithm or SAT solvers. Intractable sizes are marked by '-'.

[5]Recall that $d(Z,T) \leq B$ implies $\mathbb{P}_{G,\sigma}(Z > 0) \in \left[(1-\frac{1}{e})(1-B), (1-\frac{1}{e})(1+B)\right]$.

5.1 Rings and paths

The following Theorems 7 and 8 (based on Lemma 1) give bounds on the distance between the number of PNE and a Poisson distribution in *rings* and *paths*.

THEOREM 7. *(Distance to Poisson(1) for Rings.) Consider a random payoff GG with a ring support graph $G = (V,E)$ (that is: $V = \{1,\ldots,n\}$ and $E = \{\{i,i+1\}\}_{1\leq i\leq n} \cup \{\{n,1\}\}$) and with σ strategies per player. We have:*

$$d(Z,T) \quad \leq \quad n\sigma^{-2} + 2(\cosh(n\sigma^{-1}) - 1)$$

THEOREM 8. *(Distance to Poisson(1) for Paths.) Consider a random payoff GG with a path support graph $G = (V,E)$ (that is: $V = \{1,\ldots,n\}$ and $E = \{\{i,i+1\}\}_{1\leq i\leq n-1}$) and with σ strategies per player. We have:*

$$d(Z,T) \quad \leq \quad 2\sigma^{-1} + n\sigma^{-2} + 2(\exp(n\sigma^{-1}) - 1)$$

We compare the combinatoric bounds of Theorems 7 and 8 (see the proofs in Appendix B) to an experimental estimation of $\mathbb{P}_{G,\sigma}(Z > 0)$ (Table 1). The theoretical bounds seem to *really overestimate* the distances between Z and a Poisson(1) behavior for $\mathbb{P}_{G,\sigma}(Z > 0)$. Therefore, the simple existence of these bounds seems to be already a good indicator of a Poisson(1) behavior. In rings and paths, a growth on the number σ of strategies-per-player appears to slow down the convergence, as n tends to infinity, of $\mathbb{P}_{G,\sigma}(Z > 0)$ to 0, as indicated by Theorems 7 and 8.

5.2 Binary trees and Halins

Table 2: Estimations of the likelihood of PNE existence $\mathbb{P}_{G,\sigma}(Z > 0)$ in binary trees and Halin graphs

	binary trees			Halin		
$\sigma \setminus n$	15	31	63	15	31	63
2	0.46	0.22	0.04	0.48	0.24	0.04
3	0.46	0.16	0.04	0.50	0.35	0.10
4	0.41	0.21	0.01	0.59	0.44	0.19
5	0.46	0.17	0.08	0.64	0.49	0.24

Paths and rings differ by the two leaves, their disparitions making $\mathbb{P}_{G,\sigma}(Z > 0)$ grow faster when σ grows (Table 1). Similarly, binary trees and Halin graphs only differ by the degrees of the $(n+1)/2$ leaves. As σ grows, the likelihood $\mathbb{P}_{G,\sigma}(Z > 0)$ grows faster in Halins than in binary trees (Table 2). The negative impact on the convergence of $\mathbb{P}_{G,\sigma}(Z > 0)$ to 0 as n tends to infinity, is stronger in Halins. Poorly connected leaf-players can create instabilities.

5.3 Square-grids

Table 3: Bounds for the likelihood of PNE existence $\mathbb{P}_{G,\sigma}(Z > 0)$ in square grids

	inf	sup	inf	sup	inf	sup
$\sigma \setminus n$	16		25		36	
6	0.271	0.993	0.185	-	-	-
8	0.472	0.793	0.450	0.814	0.276	0.988
10	0.543	0.721	0.535	0.729	0.459	0.806
12	0.576	0.688	0.572	0.692	0.532	0.732
14	0.594	0.671	0.592	0.673	0.569	0.696
16	0.604	0.660	0.603	0.661	0.589	0.676

Table 1: Estimations of the probability of PNE in rings and paths, and the intervals of Theorems 7 and 8

Rings	inf	P	sup	inf	P	sup	inf	P	sup	inf	P	sup	inf	P	sup
$\sigma \setminus n$		4			8			12			16			20	
6	0.562	0.63	0.702	0.140	0.53	-	-	0.50	-	-	0.47	-	-	0.33	-
10	0.607	0.62	0.657	0.455	0.64	0.809	0.099	0.60	-	-	0.55	-	-	0.45	-
14	0.619	0.68	0.645	0.542	0.69	0.722	0.361	0.61	0.904	0.070	0.66	-	-	0.60	-
18	0.624	0.63	0.640	0.577	0.66	0.687	0.468	0.67	0.796	0.294	0.67	0.970	0.050	0.58	-
Paths		4			8			12			16			20	
6	-	0.65	-	-	0.51	-	-	0.36	-	-	0.36	-	-	0.29	-
10	0.095	0.68	-	-	0.55	-	-	0.52	-	-	0.47	-	-	0.41	-
14	0.255	0.67	-	-	0.63	-	-	0.49	-	-	0.46	-	-	0.48	-
18	0.341	0.69	-	0.005	0.58	-	-	0.60	-	-	0.55	-	-	0.51	-

We evaluate in square-grids the bounds from Lemma 1 by computing the vector $\delta_k = |\{W \in \mathcal{Q}(V) | \delta(W) = k\}|$, $k \in [0, n]$. It enables to evaluate the bounds [inf, sup], for whatever σ, at a same computational cost (Table 3). As indicated by the experimental evidences (Figure 2, Table 3), the number σ of strategies-per-player has a strong impact on the likelihood of a PNE: a small growth on σ seems to almost negate the convergence of $\mathbb{P}_{G,\sigma}(Z > 0)$ to 0 as n tends to infinity, and the number of PNE becomes bounded to a Poisson(1) distribution.

6. CONCLUSIONS

This paper gives theoretical explanations about the interplay between the graphical structure, *the sizes of the strategy sets* and the probability of existence of a PNE, in games with independent random payoffs. While for two strategies per player the existence of a PNE is unlikely for many graph structures [6, 3, 10], a reasonable growth of the number of strategies makes this likelihood behave more like a Poisson(1) distribution: when the number of players goes to infinity in (σ, q, r)-connected graphs, and when the number of players is fixed in some graphs (rings, paths, square-grids).

The next perspectives are to study random support graphs (like the models from Erdös-Rényi, Barabási-Albert, and Watts-Strogatz) different numbers of strategies per player, correlated payoffs, and evolutionary solution concepts.

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APPENDIX

A. PROOF OF LEMMA 1

The proof of Lemma 1 relies on a convergence lemma to Poisson(1). By using this convergence lemma, we demonstrate the intermediate technical Lemma 3, to get rid of the probabilites. Using this intermediate technical Lemma 3, we achieve the proof of Lemma 1.

A.1 Convergence lemma to Poisson(1)

We first remind the same convergence lemma used in [3], which comes from [1]. For simplicity, it is presented with the notations of its upcoming present application.

LEMMA 2. *Let $\{Y_s\}_{s\in S}$ be a set of Bernoulli variables with $Z = \sum_{s\in S} Y_s$ such that $\mathbb{E}[Z] = 1$. For each Y_s, we define the neighborhood of dependence $D_s \subset S$ to contain (at least) all the variables Y_t from which Y_s is not independent. Finally, let T be a Poisson(1) variable and $d(Z,T)$ the total variation distance between Z and T. Then we have:*

$$d(Z,T) \leq b_1 + b_2$$

where:

$$b_1 = \sum_{s\in S}\sum_{t\in D_s} \mathbb{P}(Y_s)\mathbb{P}(Y_t)$$

$$b_2 = \sum_{s\in S}\sum_{t\in D_s\setminus\{s\}} \mathbb{P}(Y_s \cap Y_t)$$

A.2 Intermediate technical lemma

Recall that $G = (V,E)$ is a support graph and that the discrete strategy sets S_v all have the same cardinality σ. For each global strategy profile $s \in S$, the Bernoulli variable Y_s is the indicator that s is a PNE. The distribution Z describes the number of PNE with respect to $\mathbb{P}_{G,\sigma}$. We have $\mathbb{E}_{G,\sigma}[Z] = \sum_{s\in S}\mathbb{E}_{G,\sigma}[Y_s] = \sigma^n \times \sigma^{-n} = 1$ [6].

LEMMA 3. *Let $G = (V,E)$ be a graph and let σ be the number of strategies per player. With respect to $\mathbb{P}_{G,\sigma}$, let Z be the distribution of the number of PNE and let T be a Poisson(1) distribution. We have:*

$$d(Z,T) \leq \sigma^{-n}|D_0| + \sigma^{-n}\sum_{k=1}^{n-1}|D_0^k|\sigma^k$$

where:

$$D_0 = \{s \in S : \exists v \in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\}$$

is a function of G and σ, and so is:

$$D_0^k = \left\{ s \in S\setminus\{0\} : \begin{array}{c} \forall v \in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)} \Rightarrow s_v = 0_v \\ |\{v \in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\}| = k \end{array} \right\}$$

We know no intuitions about the subsets $D_0, D_0^k \subset S$, they are technical.

PROOF. *Proof of the Lemma 3.* Now, in order to use the Lemma 2, for each $s \in S$, we need to define the "neighborhoods of dependence" of Y_s, denoted by D_s.

First, remark that for a given player v and two global action profiles s and t, if $s_{\mathcal{N}(v)} \neq t_{\mathcal{N}(v)}$, then the best response indicators $X_{s,v}$ and $X_{t,v}$ are independent. Similarly, if for all players v we have $s_{\mathcal{N}(v)} \neq t_{\mathcal{N}(v)}$, then Y_s and Y_t are independent. So, for a PNE indicator Y_s, it is sufficient for Lemma 2, to define this neighborhood of dependence:

$$D_s = \{t \in S : \exists v \in V, s_{\mathcal{N}(v)} = t_{\mathcal{N}(v)}\}$$

Remark that the sets D_s have the same size, whatever $s \in S$. Let us denote $0 \in S$ one fixed global strategy profile. Then, using Lemma 2, we have:

$$b_1 = \sum_{s\in S}\sum_{t\in D_s}\mathbb{P}(Y_s)\mathbb{P}(Y_t) = \sigma^n|D_0|\sigma^{-n}\sigma^{-n} = \sigma^{-n}|D_0|$$

Now let us proceed with b_2. Remark that:

$$\begin{aligned} b_2 &= \sum_{t\in S}\sum_{s\in D_t\setminus\{t\}} \mathbb{P}_{G,\sigma}(Y_s \cap Y_t) \\ &= \sum_{t\in S}\mathbb{P}_{G,\sigma}(Y_t)\sum_{s\in D_t\setminus\{t\}}\mathbb{P}_{G,\sigma}(Y_s|Y_t) \\ &= \sigma^{-n}\sum_{t\in S}\sum_{s\in D_t\setminus\{t\}}\mathbb{P}_{G,\sigma}(Y_s|Y_t) \qquad (1) \end{aligned}$$

For a term in the sum 1, that is for a fixed $t = 0$ in S, we have by the definition of Y_s and by the independence of the players' preferences (even conditionally to Y_0):

$$\begin{aligned} \sum_{s\in D_0\setminus\{0\}}\mathbb{P}_{G,\sigma}(Y_s|Y_0) &= \sum_{s\in D_0\setminus\{0\}}\mathbb{P}_{G,\sigma}(\cap_{v\in V}X_{s,v}|Y_0) \\ &= \sum_{s\in D_0\setminus\{0\}}\prod_{v\in V}\mathbb{P}_{G,\sigma}(X_{s,v}|Y_0) \ (2) \end{aligned}$$

For a term in the sum 2, that is for a fixed $s \in D_0 \setminus \{0\}$ and for a fixed $v \in V$, we have exactly one of the following (about $\mathbb{P}_{G,\sigma}(X_{s,v}|Y_0)$):

- If $s_{\mathcal{N}(v)} \neq 0_{\mathcal{N}(v)}$ then Y_0 gives no information and then $\mathbb{P}_{G,\sigma}(X_{s,v}|Y_0) = \mathbb{P}_{G,\sigma}(X_{s,v}) = \sigma^{-1}$.
- If $s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}$ and $s_v \neq 0_v$, since best responses are singletons, then $\mathbb{P}_{G,\sigma}(X_{s,v}|Y_0) = 0$.
- If $s_{\mathcal{M}(v)} = 0_{\mathcal{M}(v)}$, then $\mathbb{P}_{G,\sigma}(X_{s,v}|Y_0) = 1$.

To summarize:

$$\begin{aligned} &\sum_{s\in D_0\setminus\{0\}}\prod_{v\in V}\mathbb{P}_{G,\sigma}(X_{s,v}|Y_0) \\ =& \sum_{s\in D_0\setminus\{0\}}\mathbb{I}\left\{\begin{array}{c}\forall v\in V\\ s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\\ \Rightarrow s_v = 0_v\end{array}\right\}\sigma^{|\{v\in V: s_{\mathcal{N}(v)}=0_{\mathcal{N}(v)}\}|-n} \end{aligned}$$

That is why we refine the notation $D_0 = \{s \in S : \exists v \in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\}$ by introducing the notations D_0^k for $k = 1,\ldots,n$, defined by:

$$D_0^k = \left\{ s \in S\setminus\{0\} : \begin{array}{c} \forall v \in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)} \Rightarrow s_v = 0_v \\ |\{v \in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\}| = k \end{array} \right\}$$

and then we have:

$$\sum_{s\in D_0\setminus\{0\}}\mathbb{P}_{G,\sigma}(Y_s|Y_0) = \sigma^{-n}\sum_{k=1}^{n-1}|D_0^k|\sigma^k$$

And generalizing by symmetry, to any $t \in S_V$:

$$\begin{aligned} b_2 &= \sigma^{-n}\sum_{t\in S_V}\sum_{s\in D_t\setminus\{t\}}\mathbb{P}_{G,\sigma}(Y_s|Y_t) \\ &= \sigma^{-n}\sigma^{+n}\sum_{k=1}^{n-1}|D_0^k|\sigma^{k-n} \end{aligned}$$

[6] Recall that the mathematical expectation of a random variable is not always a good indicator of its "center".

which completes the proof of Lemma 3. $\square$

A.3 Achieving the proof of Lemma 1

Recall that by Lemma 3, we have:

$$d(Z,W) \leq \sigma^{-n}|D_0| + \sigma^{-n}\sum_{k=1}^{n-1}|D_0^k|\sigma^k$$

First, to upper-bound $|D_0|$, we cover the set D_0 roughly:

$$D_0 \subseteq \bigcup_{v\in V}\{s \in S : s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\}$$

which lets us upper-bound $|D_0|$ by $\sum_{v\in V}\sigma^{n-\delta(v)}$ and $\sigma^{-n}|D_0|$ by $\sum_{v\in V}\sigma^{-\delta(v)}$. We now proceed with D_0^k. Recall that:

$$D_0^k = \left\{ s \in S\setminus\{0\} : \begin{array}{c} \forall v\in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)} \Rightarrow s_v = 0_v \\ |\{v\in V, s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}\}| = k \end{array} \right\}$$

We cover D_0^k by identifying the subsets $W \subset V$ of k vertices such that $s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}$. Remark that by the definition of D_0^k, for all vertices v in one of these subsets $W \subset V$, since $s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}$ then $s_v = 0_v$. Then we can cover D_0^k by:

$$D_0^k \subseteq \bigcup_{W\in\mathcal{P}_k(V)}\{s\in S\setminus\{0\} : s_{\mathcal{M}(W)} = 0_{\mathcal{M}(W)}\}$$

Then, since we lose $|\mathcal{M}(W)| = \delta(W)+k$ degrees of freedom for $s \in S\setminus\{0\}$, we have for $1 \leq k \leq n-1$:

$$|D_0^k| \quad \leq \quad \sum_{W\in\mathcal{P}_k(V)}^{\mathcal{M}(W)\neq V}\sigma^{n-(\delta(W)+k)}$$

Summing completes the proof of Lemma 1, *weak version*:

$$\sigma^{-n}\sum_{k=1}^{n-1}|D_0^k|\sigma^k \quad \leq \quad \sum_{k=1}^{n-1}\sum_{W\in\mathcal{P}_k(V)}^{\mathcal{M}(W)\neq V}\sigma^{-\delta(W)}$$

For the *strong version* of Lemma 1, we need to consider the constraints $s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)} \Rightarrow s_v = 0_v$ of D_0^k. Given one of these $W \in \mathcal{P}_k(V)$, if a vertex v has its neighborhood $\mathcal{N}(v)$ included in $\mathcal{M}(W)$, then $s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}$, and the vertex v turns out to be one of the exactly k vertices such that $s_{\mathcal{N}(v)} = 0_{\mathcal{N}(v)}$. Therefore v is necessarily in W. Let us define $\mathcal{Q}_k(V) \subset \mathcal{P}_k(V)$ to be:

$$\mathcal{Q}_k(V) = \left\{ W \subset V : \begin{array}{c} |W| = k \ \wedge \ \mathcal{M}(W) \neq V \wedge \\ \forall v\in V,\ \mathcal{N}(v) \subseteq \mathcal{M}(W) \Rightarrow v \in W \end{array} \right\}$$

Then, we have for $1 \leq k \leq n-1$:

$$|D_0^k| \quad \leq \quad \sum_{W\in\mathcal{Q}_k(V)}\sigma^{n-(\delta(W)+k)}$$

Since $\mathcal{Q}(V) = \bigcup_{1\leq k\leq n-1}\mathcal{Q}_k(V)$, summing over k achieves the proof of Lemma 1, *strong version*. $\square$

B. PROOFS OF THE THEOREMS 7 AND 8

Proof of Theorem 7. In order to prove Theorem 7, recall that by Lemma 1 (strong version), we have:

$$d(Z,T) \leq \sum_{v\in V}\sigma^{-\delta(v)} + \sum_{W\in\mathcal{Q}(V)}\sigma^{-\delta(W)}$$

Since we are in a ring, $\sum_{v\in V}\sigma^{-\delta(v)}$ is evaluated to $n\sigma^{-2}$. For the second quantity, given W in $\mathcal{Q}(V)$, let $\gamma(W)$ be the number of connected components (CC) of $G(W)$, and let $\mathcal{Q}^{(p)}(V) = \{W \in \mathcal{Q}(V)|\gamma(W) = p\}$ be the elements of $\mathcal{Q}(V)$ with $\gamma(W) = p$ connected components. Since each CC (the black vertices in Figure 3) induces two neighbors in the ring (the green and red vertices in Figure 3) and $\delta(W) = 2\gamma(W)$, the second quantity can be upper-bounded by:

$$\sum_{W\in\mathcal{Q}(V)}\sigma^{-\delta(W)} = \sum_{p=1}^{n-1}\left|\mathcal{Q}^{(p)}(V)\right|\sigma^{-2p} = \sum_{p=1}^{\lceil n/5\rceil}2\binom{n-3p}{2p}\sigma^{-2p}$$

In the right-hand of this equality, we enumerate the cardinalities $\left|\mathcal{Q}^{(p)}(V)\right|$ by choosing $2p$ alternating green/red separators among the vertices, to build the CCs of W, as in Figure 3 (green opens a CC, and red closes it). Each CC contains at least one vertex, and since $\mathcal{N}(v) \subseteq \mathcal{M}(W) \Rightarrow v \in W$ then we must have at least two vertices between a red-closing and a green-opening. Therefore, the choice of separators can be done in an abstract ring of $n-3p$ vertices, and $\left|\mathcal{Q}^{(p)}(V)\right| = 2\binom{n-3p}{2p}$. The enumeration is done twice, to allow CCs containing the vertices 1 or n (ie: reds open and greens close). A calculus: $\binom{n-3p}{2p}\sigma^{-2p} \leq \frac{(n\sigma^{-1})^{2p}}{(2p)!}$ and the power series of the cosh function concludes the proof.

Proof of Theorem 8. In this second proof, $\sum_{v\in V}\sigma^{-\delta(v)} = 2\sigma^{-1} + (n-2)\sigma^{-2}$ and the number of CCs is enumerated by four different cases, displayed on the right of Figure 3. The first is when $1 \notin W \wedge n \notin W$. It sums up to less than $\cosh(n\sigma^{-1}) - 1$. The second and third cases are when $1 \in W \wedge n \notin W$ or $1 \notin W \wedge n \in W$. They use $2p-1$ separators and sum up to less than $2\sinh(n\sigma^{-1})$. In the fourth case, when $1 \in W \wedge n \in W$, there are at least $p \geq 2$ CCs, and this case is therefore bounded by $\cosh(n\sigma^{-1}) - 1$.

$$\begin{aligned} d(Z,T) \quad \leq \quad & 2\sigma^{-1} + (n-2)\sigma^{-2} \\ + \quad & \sum_{p=1}^{n-1}\binom{n-p-2(p-1)}{2p}\sigma^{-2p} \\ + \quad & 2\sum_{p=1}^{n-1}\binom{n-p-2(p-1)}{2p-1}\sigma^{-(2p-1)} \\ + \quad & \sum_{p=2}^{n-1}\binom{n-p-2(p-1)}{2p-2}\sigma^{-(2p-2)} \end{aligned}$$

The equality $\exp(x) = \cosh(x) + \sinh(x)$ ends this proof. $\square$

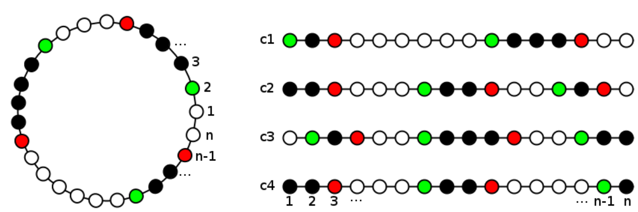

Figure 3: Combinations of separators in rings and paths: the light-green neighbors open black connected components of W, which are closed by the dark-red neighbors.