Fair Allocation of Indivisible Goods to Asymmetric Agents (Extended Abstract)

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ABSTRACT

We study fair allocation of indivisible goods to agents with *unequal entitlements*. Our emphasis is on the case where the goods are indivisible and agents have unequal entitlements. This problem is a generalization of the work by Procaccia and Wang [14] wherein the agents are assumed to be symmetric. We show that, in some cases with n agents, no allocation can guarantee better than $1/n$ approximation of a fair allocation when the entitlements are not necessarily equal. Furthermore, we devise a simple algorithm that ensures a $1/n$ approximation guarantee.

Next, we assume that the valuation of every agent for each good is bounded by the total value he wishes to receive in a fair allocation. We show it enables us to find a 1/2 approximation fair allocation via a greedy algorithm. Finally, we run some experiments on real-world data and show that, in practice, a fair allocation is likely to exist. We also support our experiments by showing positive results for two stochastic variants of the problem, namely *stochastic agents* and *stochastic items*. (The full version of the paper is available in https://arxiv.org/abs/1703.01649.)

CCS Concepts

•**Computing methodologies → Multi-agent systems;**

Keywords

fairness; indivisible; entitlements; proportionality; approximation; stochastic

1. INTRODUCTION

In this work, we conduct a study of *fairly* allocating *indivisible goods* among n agents with unequal claims on the goods. Fair allocation is a very fundamental problem that has received attention in both Computer Science and Economics. This problem dates back to 1948 when Steinhaus [17] introduced the *cake cutting* problem as follows: given n agents with different valuation functions for a cake, is it possible to divide the cake between them in such a way that every agent receives a piece whose value to her is at least $1/n$ of the whole cake? Steinhaus answered this question in the affirmative by proposing a simple and elegant algorithm which is called *moving knife*. Although this problem admits a straightforward solution, several ramifications of the cake cutting problem have been studied since then, many of which have not been settled after decades [4, 15, 7, 13, 10, 8, 18, 6, 1]. For instance, a natural generalization of the problem in which we discriminate the agents based on their entitlements is still open. In this problem, every agent claims an entitlement e_i to the cake such that $\sum e_i = 1$, and the goal is to cut the cake into disproportional pieces and allocate them to the agents such that every agent a_i's valuation for his piece is at least e_i fraction of his valuation for the entire cake. For two agents, Brams *et al.* [3] showed that at least two cuts are necessary to divide the cake between the agents. Furthermore, Robertson *et al.* [16] proposed a modified version of cut and choose method to divide the cake between two agents with portions e_1, e_2, where e_1 and e_2 are real numbers. McAvaney, Robertson, and Web [12] considered the case when the entitlements are rational numbers. They used Ramsey partitions to show that when the entitlements are rational, one can make a proper division via $O(n^3)$ cuts.

Another line of research is focused on the fair allocation of indivisible goods. In this problem instead of a heterogeneous cake, we have a set $\mathcal{M}$ of indivisible goods and we wish to distribute them among n agents. Indeed, due to trivial counter-examples in this setting, a proportional allocation is impossible to deliver. To alleviate this problem, Budish [5] proposed a concept of fairness for the allocation of indivisible goods namely *the maxmin share*. Suppose we ask an agent a_i to divide the items between the agents in a way that *he thinks* is fair to everybody. Of course, agent a_i does not take into account other agents' valuations and only incorporates her valuation function in the allocation. Based on this, we define MMS_i equal to the minimum profit that any agent receives in this allocation, according to a_i's valuation function. Obviously, in order to maximize MMS_i, agent a_i chooses an allocation that maximizes the minimum profit of the agents.

It is easy to see that MMS_i is the best possible guarantee that one can hope to obtain in this setting. If all agents have the same valuation function, then at least one of the agents receives a collection of items that are worth no more than MMS_i to her. A natural question that emerges here is

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whether a fair allocation with respect to MMS_i's is always possible? Although the experiments are in favor of this conjecture, Procaccia and Wang [14] (EC'14) refuted this by an elegant and delicate counter-example. They show such a fair allocation is impossible in some cases, even when the number of agents is limited to 3. On the positive side, they show that there always exists an allocation in which every agent's profit is at least $2/3\mathsf{MMS}_i$.

Although it is natural to assume the agents have equal entitlements, in most real-world applications, agents have unequal entitlements on the goods. For instance, in various religions, cultures, and regulations, the distribution of the inherited wealth is unequal. Furthermore, the division of mineral resources or international waters between the neighboring countries is often made unequally based on the geographic, economic, and political status of the countries.

In this paper, we study fair allocation of indivisible items with different entitlements. Our fairness criterion mimics the general idea of Budish for defining maxmin shares. Similar to Budish's proposal, in order to define a maxmin share for an agent a_i, we ask the following question: how much benefit does agent a_i expect to receive from a fair allocation, if we were to divide the goods *only based on her valuation function*? If agent a_i expects to receive a profit of p from the allocation, then she should also recognize a minimum profit of $p \cdot e_j/e_i$ for any other agent a_j, so that her own profit per entitlement is a lower bound for all agents. Thus, a fair answer to this question is the maximum value of p for which there exists an allocation such that agent a_i's profit-per-entitlement can be guaranteed to all other agents (according to her own valuation function). We define the maxmin shares of the agents based on this intuition.

1.1 Our Results and Techniques

This study is a generalization of the problem suggested by Budish in [5]. In that work, Budish proposed a new concept of fairness for indivisible items, namely the maxmin share.

Given a set $\mathcal{N}$ of n agents and a set $\mathcal{M}$ of m items, for every agent a_i and set S of items, we denote by $V_i(S)$ the total value of the items in S to a_i. We assume that the valuation functions are additive, that is $V_i(S) = \sum_{b_j \in S} V_i(\{b_j\})$.

Also, let $e_1, e_2, \ldots, e_n$ be n real numbers between 0 and 1 with the property that $\sum_i e_i = 1$. For every agent a_i, e_i denotes her claim on the goods. Fix an agent a_i and let $A = B_1, B_2, \ldots, B_n$ be a partitioning of $\mathcal{M}$ into n bundles. Define the fairness of an allocation A as

$$F_A^i = \min_j \frac{V_i(A_j)}{V_i(\mathcal{M})e_j} \tag{1}$$

Let $A^* = \langle A_1^*, A_2^*, \ldots, A_n^* \rangle$ be an allocation by a_i that maximizes $F_{A^*}^i$. The weighted maxmin share of agent a_i is defined in the same way as:

$$\mathsf{WMMS}_i = F_{A^*}^i V_i(\mathcal{M}) e_i = e_i \min_j \frac{V_i(A_j^*)}{e_j}$$

For a more intuitive description of the criteria,we refer the reader to the full version of the paper. For brevity, we use WMMS_i to refer to $\mathsf{WMMS}_i(\mathcal{M})$, when other parameters are clear from the context. Note that the maxmin share notion defined in [5] is a special case of our definition, when $e_i = 1/n$ for all a_i. For a real number $0 < \alpha \leq 1$, we call an allocation of $\mathcal{M}$ to the players α-WMMS, if the total value of the items allocated to every agent a_i is at least $\alpha\mathsf{WMMS}_i$ to her.

Our first result is regarding the existence of approximately fair allocations. We show in some cases, no allocation is better than $1/n$-WMMS.

THEOREM 1.1. *No algorithm can guarantee any allocation better than $1/n$-WMMS. Moreover, a $1/n$-WMMS allocation can be obtained via a greedy algorithm.*

We show Theorem 1.1 via a counter-example. In this example, we have $n-1$ agents with very small entitlements (ϵ) and a monopolist agent whose entitlement is $1-(n-1)\epsilon$. We show in this example that there is set of items which a monopolist agent cannot obtain a value more than $1/n + n\epsilon$ from her allocated items unless one of the other agents receives a set of items whose value to her is zero. Therefore, no allocation can be better than $1/n$-WMMS.

We complement this impossibly result by an algorithm that ensures a $1/n$ guarantee. Our proposal is very simple and easy to implement: we allocate the items to the agents in turns. In every agent a_i's turn, we ask her to collect her most favorite item from the remaining items. In this algorithm, we start from the agent with the highest entitlement, and continue on to other agents in descending order of their entitlements. We repeat this procedure until no items are left.

A closer look at our counter-example reveals a very unnatural and unrealistic structure. Every agent with an ϵ entitlement, has a valuation of $1-(n-1)\epsilon$ for an item. In other words, an agent whose entitlement is small has a high valuation for an item. One natural restriction to avoid these situations is to assume $V_i(\{b_j\}) \leq \mathsf{WMMS}_i$ for every agent a_i and item b_j. Note that this assumption is w.l.o.g for the symmetric case.

THEOREM 1.2. *There exists an algorithm that finds a $1/2$-WMMS allocation for the problem when $V_i(\{b_j\}) \leq \mathsf{WMMS}_i$ for every agent a_i and item b_j.*

When the entitlements of the agents are all equal to $1/n$, providing a $1/2$ guarantee is quite straightforward. For the case of unequal entitlements however, the algorithm fails to provide any guarantees. For this case, we propose an algorithm guaranteeing a $1/2$-WMMS allocation.

Finally, we run some experiments on real-world data and show WMMS allocations are likely to exist in practice. We support these experiments by studying the stochastic variants of the problem. Our focus is on two models namely *stochastic agents* and *stochastic items*. In the stochastic agents setting, we assume every agent has a distribution of valuations, and her value for an item is drawn independently from her distribution. This model has been studied in a series of previous works [5, 11, 2, 9]. In the stochastic items model, we assume every item has a distribution of valuations, and every agent's valuation for that item is drawn from the corresponding distribution. We show in both models, as the number of items increases, a WMMS allocation is more likely to exist. This observation aligns with our experiments.

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