Consensus on Social Graphs under Increasing Peer Pressure

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ABSTRACT

In this paper, we present a novel generalized framework for expressing peer influence dynamics over time in a set of connected individuals, or agents. The proposed framework supports the representation of individual variability through parametrization accounting for differences in susceptibility to peer influence and pairwise relationship strengths. Modeling agents' opinions and behaviors as strategies changing discretely and simultaneously, we formally describe the evolution of strategies in a social network as the composition of contraction maps. We identify points of convergence and analyze these points under various conditions.

DISCUSSION

In a connected social network, we show that agents' strategies converge to the optimal solution, namely consensus, if and only if peer pressure increases without bound. If the peer pressure is bounded and individual agents fail to change their opinions in the presence of dissonance with their peers, then these agents will not converge to an efficient distribution.

Our notion of persuasion and peer-pressure is related to the psychology literature on belief formation and social influence. We follow Friedkin's foundational theory [1] that strong ties are more likely to affect users' opinions and result in persuasion or social influence. Underpinning our model is also the notion of mimicking. Brewer and more recently Van Bareen [2, 3] suggest that mimicking is used when individuals feel out of a group and therefore will alter their behavior (to a point [3]) to be more socially accepted.

Assume that the agents' network is represented by a simple graph $G = (V, E)$ where vertexes V are users (or agents), and edges E are the social (communications) connections between them. It is clear that disconnected sections of the graph are independent, so we assume that G is connected. For the remainder of the paper, let $V = \{1, 2, \dots n\}$, so E is a subset of the two-element subsets of V. Assume we are given a set of non-negative vertex weights s_i and positive edge weights w_{ij} respectively for $i, j \in V$. In addition, we assume at least one vertex weight is positive. Without loss of generality, assume the range of opinions to be the interval $[0, 1]$ and each user has a private constant preference $x_i^+ \in [0, 1]$. The user's opinion value at time k is $x_i^{(k)} \in [0, 1]$. The set of all such values are denoted by the vector $\mathbf{x}^{(k)}$ while the set of constant private preferences is $\mathbf{x}^+$.

Each agent selects its next choice by minimizing an objective function representing social stress:

$$J_i(x_i^{(k)}, \mathbf{x}^{(k-1)}, k) = s_i \left(x_i^{(k)} - x_i^+\right)^2 + \rho^{(k)} \sum_{j=1}^{n} w_{ij} \left(x_i^{(k)} - x_j^{(k-1)}\right)^2$$

Here: $s_i(x_i^{(k)} - x_i^+)^2$ is the internal stress felt by User i as a result of deviations from her preferred state x_i^+. The quantity:

$$\rho^{(k)} \sum_{j=1}^{n} w_{ij} \left(x_i^{(k)} - x_j^{(k-1)}\right)^2,$$

is the social stress experienced by User i as a result of deviations from her peers. In Expression , $\rho^{(k)}$ is the peer pressure factor, denoting how strongly others' previous opinions influence person i. The more pressure there is to come to a consensus, the larger $\rho^{(k)}$ becomes, and therefore disagreement causes more stress.

Under these assumptions, the first order necessary conditions are sufficient for minimizing $J_i(x_i^{(k)}, \mathbf{x}^{(k-1)}, k)$ and after differentiation we solve:

$$s_i \left(x_i^{(k)} - x_i^+\right) + \rho^{(k)} \sum_{j=1}^{n} w_{ij} \left(x_i^{(k)} - x_j^{(k-1)}\right) = 0$$

Let $d_i = \sum_{j=1}^{n} w_{ij}$ be the weighted degree of vertex i, then:

$$x_i^{(k)} = \frac{s_i x_i^+ + \rho^{(k)} \sum_{j=1}^{n} w_{ij} x_j^{(k-1)}}{s_i + \rho^{(k)} \sum_{j=1}^{n} w_{ij}} \tag{1}$$

minimizes $J_i(x_i^{(k)}, \mathbf{x}^{(k-1)}, k)$.

Let $\mathbf{A}$ be the $n \times n$ weighted adjacency matrix of G. In addition, let $\mathbf{D}$ be the $n \times n$ matrix with d_i on the diagonal and let $\mathbf{S}$ be the $n \times n$ matrix with s_i on the diagonal. Using these terms, the above recurrences can be written as:

$$\mathbf{x}^{(k)} = \left(\mathbf{S} + \rho^{(k)}\mathbf{D}\right)^{-1} \left(\mathbf{S}\mathbf{x}^+ + \rho^{(k)}\mathbf{A}\mathbf{x}^{(k-1)}\right).$$

Consensus Convergence: We say that the agents converge to consensus $\mathbf{x}^*$ if there is some N so that for all $n > N$, $\|\mathbf{x}^* - \mathbf{x}^{(n)}\| < \epsilon$ for some small $\epsilon > 0$. This represents meaningful compromise on the issue under consideration.

Appears in: *Proceedings of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2017), S. Das, E. Durfee, K. Larson, M. Winikoff (eds.), May 8–12, 2017, São Paulo, Brazil.*

Our analysis requires two lemmas, which are instrumental for our convergence results. The first is proved in Chapter 13 [4]. Proofs of all remaining results can be found in [5].

Lemma 1. *If $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the weighted graph Laplacian, then $\mathbf{L}$ has a eigenvalue 0 with multiplicity 1 and a corresponding eigenvector $\mathbf{1}$ where $\mathbf{1}$ is the vector of all 1's.* □

Lemma 2. *For any $\rho^{(k)} > 0, \mathbf{S} + \rho^{(k)}\mathbf{L}$ is invertible.*

We now view each step of the evolutionary process as a function mapping the previous opinion vector to the next one. These functions $F_k : [0, 1]^n \to [0, 1]^n$ can be defined as follows:

$$F_k(\mathbf{x}) = \left(\mathbf{S} + \rho^{(k)}\mathbf{D}\right)^{-1}\left(\mathbf{S}\mathbf{x}^+ + \rho^{(k)}\mathbf{A}\mathbf{x}\right)$$

and let:

$$G_k = F_k \circ F_{k-1} \circ \cdots \circ F_1$$

Then $\mathbf{x}^{(k)} = F_k(\mathbf{x}^{(k-1)})$ and $\mathbf{x}^{(k)} = G_k(\mathbf{x}^{(0)})$. That is, iterating these F_k captures the evolution of $\mathbf{x}^{(k)}$. We argue that for each k, F_k is a contraction and therefore has a fixed point by the Banach Fixed Point Theorem [6].

Theorem 3. *For all k, F_k is a contraction map with fixed point given by $\overline{\mathbf{x}}^{(k)} = (\mathbf{S} + \rho^{(k)}\mathbf{L})^{-1}\mathbf{S}x^+$.*

The following lemma allows us to consider the matrices $(\mathbf{S}+\rho^{(k)}\mathbf{L})^{-1}$ for $k \in \{1, 2, \dots\}$ in $\mathrm{GL}_n(\mathbb{R})$ (the Lie group of invertible $n\times n$ real matrices) as perturbations. This enables effective approximations of asymptotic behaviors.

Lemma 4. *Let $\{\mathbf{b}_1 \dots \mathbf{b}_n\}$ an orthonormal basis of $\mathbb{R}^n$. Also let $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible symmetric linear transformation (invertible square matrix) and $\{\mathbf{u}_1, \dots \mathbf{u}_n\}$ be a set of unit vectors such that for a small constant δ, $\mathbf{M}^{-1}\mathbf{b}_1 = \lambda\mathbf{b}_1 + O(\delta)\mathbf{u}_1$ and $\mathbf{M}^{-1}\mathbf{b}_j = O(\delta)\mathbf{u}_j$ for $j \neq 1$.*

Then if $\|\mathbf{v}\| = 1$, and $s \in \mathbb{R}$, then unless $(\mathbf{M} + s\mathbf{v}\mathbf{v}^T)$ is not invertible, there exists a set of unit vectors $\{\mathbf{u}'_1, \dots \mathbf{u}'_n\}$ such that $(\mathbf{M} + s\mathbf{v}\mathbf{v}^T)^{-1}\mathbf{b}_1 = \frac{\lambda}{1+s\lambda(\mathbf{v}^T\mathbf{b}_1)^2}\mathbf{b}_1 + O(\delta)\mathbf{u}'_1$ and $(\mathbf{M} + s\mathbf{v}\mathbf{v}^T)^{-1}\mathbf{b}_j = O(\delta)\mathbf{u}'_j$ for $j \neq 1$.

The result is based on the Sherman-Morrison formula. In proving Theorem 5, we establish an instance of the necessary conditions of this lemma. Thus the lemma is not vacuous.

Theorem 5. *If $\lim_{k\to\infty} \rho^{(k)} = \infty$, then:*

$$\lim_{k\to\infty} \overline{\mathbf{x}}^{(k)} = \frac{\sum_{i=1}^n s_i x_i^+}{\sum_{i=1}^n s_i}\mathbf{1}.$$

Since peer pressure increases in each step, no single F_k is sufficient to model the process of convergence. We need to show that this is the attractive fixed point of the entire process. From Theorem 2 of [7] and the fact that F_k are contractions whose fixed points converge, we have:

Theorem 6. *Let $G_k = F_k \circ G_{k-1} = F_k \circ F_{k-1} \circ \cdots \circ F_1$ for each $k \geq 0$. Then $G = \lim_{k\to\infty} G_k$ is a constant function with value $\lim_{k\to\infty} \overline{\mathbf{x}}^{(k)}$, and the convergence is uniform.*

In the case of increasing but bounded peer pressure, we have:

$$\lim_{k\to\infty} \rho^{(k)} \leq \rho^*$$

Further, this limit always exists by monotone convergence. Intuitively, this means the influence of others is limited, and that personal preferences will always slightly skew the opinions of others. Again, this is consistent with social influence theories on bounded peer pressure and trade-offs with comfort level [3].

Theorem 7. *Suppose $\rho^{(k)}$ is increasing and bounded and that:*

$$\lim_{k\to\infty} \rho^{(k)} = \rho^*,$$

then

$$\lim_{k\to\infty} \mathbf{x}^{(k)} = (\mathbf{S} + \rho^*\mathbf{L})^{-1}\mathbf{S}\mathbf{x}^+.$$

Having proven that convergent points exist, we can now analyze their efficiency. Define the utility of these convergent points to be the sum of the stress of the agents when the state $\mathbf{x}$ is constant. Formally let the global utility function be:

$$\begin{aligned}
U(\mathbf{x}) &= \sum_i \lim_{k\to\infty} J_i(x_i, \mathbf{x}, k) \\
&= \sum_{i=1}^n s_i(x_i - x_i^+)^2 + 2\left(\sum_{ij\in E}(x_i - x_j)^2\right)\lim_{k\to\infty}\rho^k \\
&= (\mathbf{x} - \mathbf{x}^+)^T\mathbf{S}(\mathbf{x} - \mathbf{x}^+) + 2\lim_{k\to\infty}\rho^k\mathbf{x}^T\mathbf{L}\mathbf{x} \\
&= \lim_{k\to\infty}\mathbf{x}^T(\mathbf{S} + 2\rho^k\mathbf{L})\mathbf{x} - 2\mathbf{x}^T\mathbf{S}\mathbf{x}^+ + (\mathbf{x}^+)^T\mathbf{S}\mathbf{x}^+
\end{aligned}$$

The following lemma is immediately clear from by construction of U and J_i:

Lemma 8. *The global utility function $U(\mathbf{x})$ is convex.* □

Leveraging the above Lemma on the utility function, we are now ready to present our main optimality results on the convergence point conditions.

Theorem 9. *The convergent point $\lim_{k\to\infty}\mathbf{x}^{(k)}$ minimizes utility if and only if $\lim_{k\to\infty}\rho^k = \infty$.*

Corollary 10. *The cost of anarchy is 1 if and only if $\lim_{k\to\infty}\rho^k = \infty$.* □

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