Existence and computation of equilibria of first-price auctions with integral valuations and bids

(Extended Abstract)

Guillaume Escamocher
Ecole Normale Supèrieure de
Cachan, 94235,Paris, France
gescamoc@dptinfo.ens-
cachan.fr

Peter Bro Miltersen
Aarhus University
8200 Aarhus N, Denmark
bromille@cs.au.dk

Rocio Santillan R.
Aarhus University
8200 Aarhus N, Denmark
rocio@cs.au.dk

ABSTRACT

We consider existence and computation of symmetric Pure Strategy Nash Equilibrium (PSNE) in single-item, sealed-bid, first-price auctions with integral valuations and bids. For the most general case, we show that existence of PSNE is NP-hard. Then, we present algorithmic results for the case of independent valuations and two ways of breaking ties: Vickrey tie-breaking and random tie-breaking.

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1. MAIN RESULTS

The most general setup for Bayesian single-item sealed-bid first-price auctions consists of a finite set of states M of the world. To each state i of a world and each risk-neutral bidder j is associated a valuation $v_{ij} \in \mathbb{N}$ and a signal $s_{ij} \in \mathbb{N}$ that bidder j receives and must base his bid upon. Also, each state i of the world occurs with some probability p_i. The resulting distribution is the *common prior* of the bidders. After receiving their signal, the bidders simultaneously submit bids. The highest bidder wins the auction and receives as payoff the difference between his valuation and his bid. Ties may occur. We consider two different ways of breaking these. In the Vickrey tie-breaking auction, all bidders submit an *auxiliary* bid in addition to their primary bid. In case of ties, a Vickrey (i.e., second price) auction is conducted among the bidders with the highest primary bid, using their auxiliary bids. Ties in the Vickrey auction can be broken arbitrarily. This tie-breaking rule was introduced

by Maskin and Riley [3]. In the random tie-breaking case, if k bidders all submit the highest bid, we choose one of them as the winner in a (uniform) lottery.

A pure strategy for a bidder is given by a map from signals to bids $\beta : \mathbb{N} \to \mathbb{N}$ so that the bidder submits bid $b_{ij} = \beta(s_{ij})$. A strategy profile is a family of strategies, one for each bidder. A strategy profile is a pure strategy Nash equilibrium (PSNE) if each strategy in the profile achieves the best possible expected payoff for the bidder, given that the other bidders bid according to the profile.

We prove the following theorem:

THEOREM 1. *Given an auction with random tie-breaking rule, it is* **NP***-hard to decide if it has a PSNE.*

Next, we consider the case of independent, identically distributed, private valuations (we call these p.i.i.d. auctions). These fit in the general framework as follows: The states of the world are $M = V^n$ where n is the number of bidders and V the set of valuations; the probability distribution on M is a product distribution, and $s_{ij} = v_{ij}$ for all i, j (each bidder learns his valuation and nothing else). The valuations are independent samples from a probability distribution on $V = \{v_1, v_2, \ldots, v_k\}$ with $v_1 < v_2 < \cdots < v_k$ and with probability p_i of v_i occurring. Also, we restrict attention to symmetric Nash equilibria, where all bidders use the same strategy. We denote by β the symmetric profile where all bidders use strategy β.

Given standard properties of second-price auction [1], in any symmetric PSNE of p.i.i.d. auction with Vickrey tie-breaking, all bidders submit their valuation as their auxiliary bid, so our analysis is concerned only with the primary bid.

We prove:

THEOREM 2. *A p.i.i.d auction with Vickrey tie-breaking has either no PSNE or exactly two. When two equilibria exist, they differ only by their bid for the lowest valuation, and exactly one of them is undominated.*

We algorithmically characterize the distributions for which symmetric PSNE exist for any finite support size and any finite number of bidders. Algorithm 1 returns a candidate for the undominated equilibrium. The algorithm, which has a dynamic programming flavour, is given the density table of the distribution as input and runs in time linear in the support size of the distribution. In the algorithm, we denote by $E_\Pi(v_i, b_i, \beta)$ the expected payoff of a bidder with valuation v_i if he chooses to bid b_i, while other bidders bid according

to the profile β. Also, P_i is the probability that the highest valuation among all players except one (e.g., Player 1) is v_i, i.e., $P_i = \left(\sum_{j=1}^{i} p_j \right)^{n-1} - \left(\sum_{j=1}^{i-1} p_j \right)^{n-1}$.

Algorithm 1 Algorithm for producing equilibrium for Vickrey tie-breaking auction

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1: $b_1 \leftarrow v_1 - 1$
2: $A_1(1), A_2(1) \leftarrow 0$
3: $A_3(1) \leftarrow P_1$
4: $b_2 \leftarrow v_1$
5: $A_1(2) \leftarrow P_1$
6: $A_2(2) \leftarrow 0$
7: $A_3(2) \leftarrow A_3(1) + P_2$
8: **for** $i = 3$ to k **do**
9: $\quad b_i \leftarrow b_{i-1}$
10: $\quad A_1(i) \leftarrow A_1(i-1)$
11: $\quad A_2(i) \leftarrow A_2(i-1) + (A_3(i-1) - A_1(i-1))(v_i - v_{i-1})$
12: $\quad A_3(i) \leftarrow A_3(i-1) + P_i$
13: $\quad$ **if** $E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b_i + 1, \beta)$ **then**
14: $\quad\quad b_i \leftarrow b_{i-1} + 1$
15: $\quad\quad A_1(i) \leftarrow A_3(i-1)$
16: $\quad\quad A_2(i) \leftarrow 0$

We prove that Algorithm 1 outputs a symmetric PSNE for a given p.i.i.d auction with Vickrey tie-breaking if one exists.

The algorithm produces a strategy profile even if there is no equilibrium. However, a pure strategy β can be verified to be a symmetric PSNE in quadratic time.

A crucial and interesting property of the equilibrium is monotonicity of the strategies. A pure strategy is said to be monotonic if for all $v_j < v_i$, $\beta(v_j) \leq \beta(v_i)$, i.e., $b_j \leq b_i$.

THEOREM 3. *A symmetric PSNE for a p.i.i.d auction with Vickrey tie-breaking is monotonic.*

We also prove this property for the case of random tie-breaking auctions.

LEMMA 1. *In a p.i.i.d. auction with random tie-breaking, any symmetric PSNE β is monotonic.*

In other respects, the auction with random tie-breaking is much harder to understand than the auction with Vickrey tie-breaking. In particular, we do not have an algorithm similar to Algorithm 1 for the random tie-breaking case. However, we do have a full understanding of the case of *bivalued* distributions, i.e., distributions with support size 2.

Let p be the probability that a bidder gets the lowest valuation. Let $q = \frac{1-p^n}{n(1-p)}$. Let also

$$ g(p, n) = \left\lfloor \frac{q}{p^{n-1}} \left\lfloor \frac{1}{1-q} \right\rfloor \right\rfloor + 1 $$

THEOREM 4. *Let $V = \{v_1, v_2\}$ be a bivalued distribution with $p_1 = p > 0$ and $p_2 = 1-p > 0$. Then, the corresponding p.i.i.d. auction with n bidders has a symmetric PSNE if and only if $v_2 - v_1 \leq g(p, n)$.*

We can also provide the equilibrium strategies for bivalued distributions. If $v_2 = v_1 + 1$, it is easily checked that β with $\beta(v_2) = \beta(v_1) = 1$ is a PSNE. If $v_2 \geq v_1 + 2$, we prove that $b_1 = \beta(v_1) = v_1$ and $b_2 = \beta(v_2) = \max(v_1 + 1, v_2 - \lfloor \frac{1}{1-q} \rfloor)$ is a symmetric PSNE.

It is left open how to extend this result for the case of k valuations. We present an heuristic for the case of uniform distributions on some set of integer valuations, and $n = 2$ players. The heuristic, Algorithm 2, attempts to build an equilibrium from below and empirically often produces an equilibrium when one exists on small inputs (but may fail in producing one even when one exists and may produce a strategy profile which is not an equilibrium). It is based on ideas similar to the ones used in our (correct) algorithm for the case of Vickrey tie-breaking.

Algorithm 2 Heuristic for producing an equilibrium for flat distributions, two bidders, and random tie-breaking

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1: $start = 2$
2: $b_1 \leftarrow v_1$
3: **if** $v_2 - v_1 = 7$ **then**
4: $\quad b_2 \leftarrow v_1 + 3$
5: $\quad start = 3$
6: **for** $i = start$ to k **do**
7: $\quad t \leftarrow$ number of bids that are equal to b_{i-1}
8: $\quad$ **if** $v_i - b_{i-1} > 2i + 2$ **then**
9: $\quad\quad$ **return** Fail
10: $\quad$ **else if** $v_i - b_{i-1} \geq 2i$ **then**
11: $\quad\quad b_i = b_{i-1} + 2$
12: $\quad$ **else if** $v_i - b_{i-1} \geq \frac{2i-1}{t+1}$ **then**
13: $\quad\quad b_i = b_{i-1} + 1$
14: $\quad$ **else**
15: $\quad\quad b_i = b_{i-1}$

We can prove that the heuristic is correct in a very special case.

THEOREM 5. *For any flat distribution of valuations, if we have $v_{i+1} - v_i \geq 4$ for every i, then Algorithm 2 outputs Fail if there is no symmetric PSNE for the p.i.i.d auction and otherwise outputs such a symmetric PSNE.*

A complete version of the paper can be found at [2].

2. ACKNOWLEDGMENTS

3. REFERENCES

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