Constrained Coalitional Games: Formal Framework, Properties, and Complexity Results

(Extended Abstract)

Gianluigi Greco
Dept. of Mathematics
University of Calabria
I-87036, Rende, Italy
ggreco@mat.unical.it

Enrico Malizia, Luigi Palopoli, and Francesco Scarcello
DEIS
University of Calabria
I-87036, Rende, Italy
{emalizia,palopoli,scarcello}@deis.unical.it

ABSTRACT

A general approach to define constrained coalitional games is proposed, i.e., TU games where additional, application-oriented *constraints* are imposed on the possible outcomes. It is observed that constrained games are succinct Non-Transferable (NTU) specifications, which yet retain (some of) the nice properties of the underlying TU games. In fact, a clear picture about the preservation properties of TU solution concepts is depicted, and a thorough analysis is eventually carried out, to assess the impact of issuing constraints on the computational complexity of these solution concepts.

Categories and Subject Descriptors

J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*; F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity

General Terms

Economics, Theory, Algorithms

Keywords

Coalitional Games, Computational Complexity

1. TU GAMES AND CONSTRAINTS

Cooperative game theory provides—under the concept of *coalitional games*—an elegant framework for modeling multi-agent systems where agents might collaborate with other agents by forming *coalitions*, in order to guarantee themselves some advantage. Within this framework, to each coalition $S \subseteq N$ (where N is the set of all the players, also called the grand-coalition), is assigned a certain worth $v(S)$, and the outcome is a vector of payoffs $(x_i)_{i \in N} \in \mathbb{R}^{|N|}$ that is meant to specify the distribution of the worth granted to the set of all players of the game. Coalitional games are most often classified in different species, based on the mechanisms underlying this payoff distribution.

Cite as: Constrained Coalitional Games: Formal Framework, Properties, and Complexity Results, (Extended Abstract), Gianluigi Greco, Enrico Malizia, Luigi Palopoli, Francesco Scarcello, *Proc. of 8th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2009)*, Decker, Sichman, Sierra and Castelfranchi (eds.), May, 10–15, 2009, Budapest, Hungary, pp. 1295–1296

The best known and largely studied class therein is that of coalitional games *with transferable utility* (or TU games) [3], where no constraint whatsoever is imposed over the way coalitional worths can be distributed amongst coalition members. There are cases, however, where players cannot freely distribute the coalition worth so that the TU framework is not appropriate to the situation.

Example 1.1. Three brothers, Tim, John and Jim, aged 10, 8 and 5, resp., have collected into a piggy money-box all the small Euro coins that Mom every week has given to each of them. Now the time has come to break the money-box and divide its content. In order to avoid quarrels among the kids, Mom decides that the distribution has to go with their ages, so that Tim will deserve at least 10/8 the money John will get and John, in his turn, will receive at least 8/5 of Jim's money share. The money-box gets broken and the little treasure of seven Euros and ninety Euro cents (as resulting from the available coin sets including one-hundred 1-cent coins, seventy 2-cent coins, fifty 5-cents coins, thirty 10-cent coins) can then be divided amongst the kids.

In fact, it would be useful to model this scenario through a coalitional game as to determine a "fair" way of distributing the worth amongst the brothers. However, because of Mom's rule and of the integer constraints determined by the available coins, the TU framework is not appropriate for modeling this scenario. ◁

In order to face such cases, one may define a game *without transferable utility* (short: NTU) where the worth function just consists of a list of all those allowed worth distributions (called *consequences*) associated with the members of any given coalition. However, whenever there are many consequences, explicitly listing all of them maybe either ineffective or even impossible, if their number is infinite. Our contribution is to face this issue, by investigating a general and abstract framework for coalitional games, which allows to succinctly specify non-transferable conditions on arbitrary sets of players via a set of *mixed-integer linear (in)equalities* defined over an underlying TU game.

Example 1.2. Let us consider again Example 1.1, and note that the scenario can easily be modeled by means of a set of linear (in)equalities, such that a few variables taking values from the set $\mathbb{N}$ of natural numbers are also used (in the following, we shall denote the three brothers Tim, John and Jim by using the indexes 1, 2 and 3, respectively):

$$\begin{cases} x_i = 1\times\alpha_1^i + 2\times\alpha_2^i + 5\times\alpha_5^i + 10\times\alpha_{10}^i, \forall 1\leq i\leq 3 \\ \alpha_1^1+\alpha_1^2+\alpha_1^3 = 100 \\ \alpha_2^1+\alpha_2^2+\alpha_2^3 = 70 \\ \alpha_5^1+\alpha_5^2+\alpha_5^3 = 50 \\ \alpha_{10}^1+\alpha_{10}^2+\alpha_{10}^3 = 30 \\ x_1 \geq 10/8\times x_2 \\ x_2 \geq 8/5\times x_3 \\ x_i\in\mathbb{R},\ \alpha_1^i,\alpha_2^i,\alpha_5^i,\alpha_{10}^i\in\mathbb{N}, \forall 1\leq i\leq 3 \end{cases}$$

where an auxiliary variable of the form α_j^i denotes the number of coins of value j taken by player i. ◁

Following the idea exemplified above, in the full paper we define the formal framework of constrained games, and investigate its properties. In particular, we point out the expressiveness of the framework, and we study the analytical and computational properties characterizing several well-known *solution concepts* for coalitional games. We next overview our results in these three directions.

2. REPRESENTATION POWER

Observe that constrained coalitional games are special cases of NTU games, due to their ability of restricting the space of the possible worth distributions. With respect to NTU frameworks where consequences are explicitly listed (see, e.g, [3]), it is easy to see that constrained games are "infinitely" more succinct, because constraints can compactly encode portions of $\mathbb{R}^{|N|}$. Moreover, we have shown that, even when finitely many consequences have to be specified, constrained games may be exponentially more succinct.

3. SOLUTION CONCEPTS

Despite constrained games emerged as games with non-transferable utility, they are essentially built within a TU setting. Hence, no specific and ad-hoc solution concept has to be conceived. Rather, all the concepts defined for classical TU games find an immediate counterpart when a set LC of constraints is issued, provided feasible imputations are restricted over the set $\Omega(\texttt{LC})$ of those imputations satisfying LC. The second line of research was then to analyze how constraints affect the basic properties of some of the main solution concepts defined in the literature (see, e.g, [3]), namely, *core* ($\mathscr{C}$), *kernel* ($\mathscr{K}$), *bargaining set* ($\mathscr{B}$), and *nucleolus* ($\mathscr{N}$). Figure 1 graphically summarizes our results.

In the figure, $\mathscr{S}|_{\texttt{LC}}$ denotes the constrained version of the concept $\mathscr{S}$; in addition, the fact that sets are disjoint means that in some games their intersection can be empty.

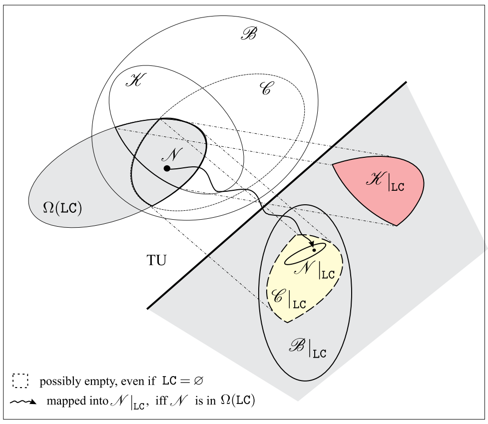

Figure 1: Relationships among Solution Concepts.

Problem	Result
Core-Check	co-**NP**-complete
Kernel-Check	$\mathbf{\Delta_2^P}$-complete
BargainingSet-Check	$\mathbf{\Pi_2^P}$-complete
Nucleolus-Check	**PP**-hard□
Core-NonEmptiness	$\mathbf{\Sigma_2^P}$-complete‡
Kernel-NonEmptiness	$\mathbf{\Sigma_2^P}$-complete°
BargainingSet-NonEmptiness	$\mathbf{\Sigma_3^P}$-complete°

Figure 2: Complexity results for constrained games. For TU games: □in $\mathbf{\Pi_2^P}$; ‡co-NP-complete; °Trivial.

The reader may notice that the portions of the core and of the kernel that satisfy all the constraints are "preserved." Instead, the bargaining set of the constrained game is basically unrelated with the bargaining set of the underlying TU game. In addition, the core is contained in the bargaining set, and whenever the core is not empty it contains the nucleolus. In particular, the nucleolus of constrained games may consist of more than one imputation, but it is guaranteed to be non-empty (provided a closeness condition, which is met in practice), and to contain the nucleolus of the underlying TU game, whenever this falls in $\Omega(\texttt{LC})$.

4. COMPUTATIONAL PROPERTIES

Finally, we assessed the impact of adding constraints on the computational complexity underlying the above solution concepts. In particular, following [1], we considered games specified in *compact form*, i.e., whose representation size is polynomial in the number of players, and we studied the problems of checking whether a given imputation satisfies the conditions needed to be in the core, in the kernel, in the bargaining set, or in the nucleolus (short: Core-Check, Kernel-Check, BargainingSet-Check, and Nucleolus-Check problems). Also, we studied the complexity of checking the emptiness of these notions (short: Core-NonEmptiness, Kernel-NonEmptiness, and BargainingSet-NonEmptiness problems), but for the nucleolus, which is always nonempty if $\Omega(\texttt{LC}) \neq \varnothing$. In particular, note that all membership results assume (polynomially) bounded precision, when real numbers are dealt with.

Figure 2 summarizes our results. Note that for Core-Check, Kernel-Check, and BargainingSet-Check intractability results have been provided even for standard TU games. Moreover, while kernel and bargaining set are "well-behaved" when constraints are added to the game, a slight increase of complexity emerged with the core, whose complexity raises one level up in the polynomial hierarchy for the non-emptiness problem (from co-**NP** in absence of constraints [2], to $\mathbf{\Sigma_2^P}$), and with the nucleolus (that becomes hard for the class **PP**, while it is confined in $\mathbf{\Pi_2^P}$ for TU games). Also note that non-emptiness problems for bargaining sets and kernel are trivial over TU games, since these solutions are always nonempty there.

5. REFERENCES

[1] J. M. Bilbao. *Cooperative Games on Combinatorial Structures*, Kluwer Academic, Boston, USA, 2000.

[2] E. Malizia, L. Palopoli, and F. Scarcello. Infeasibility certificates and the complexity of the core in coalitional games. *Proc. of IJCAI'07*, pp.1402–1407.

[3] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press, Cambridge, MA, USA, 1994.