Sustaining Cooperation on Networks: An Analytical Study based on Evolutionary Game Theory

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ABSTRACT

We analytically study the role played by the network topology in sustaining cooperation in a society of myopic agents in an evolutionary setting. In our model, each agent plays the Prisoner's Dilemma (PD) game with its neighbors, as specified by a network. Cooperation is the incumbent strategy, whereas defectors are the mutants. Starting with a population of cooperators, some agents are switched to defection. The agents then play the PD game with their neighbors and compute their fitness. After this, an evolutionary rule, or imitation dynamic is used to update the agent strategy. A defector switches back to cooperation if it has a cooperator neighbor with higher fitness. The network is said to sustain cooperation if almost all defectors switch to cooperation. Earlier work on the sustenance of cooperation has largely consisted of simulation studies, and we seek to complement this body of work by providing analytical insight for the same.

We find that in order to sustain cooperation, a network should satisfy some properties such as small average diameter, densification, and irregularity. Real-world networks have been empirically shown to exhibit these properties, and are thus candidates for the sustenance of cooperation. We also analyze some specific graphs to determine whether or not they sustain cooperation. In particular, we find that scale-free graphs belonging to a certain family sustain cooperation, whereas Erdos-Renyi random graphs do not. To the best of our knowledge, ours is the first analytical attempt to determine which networks sustain cooperation in a population of myopic agents in an evolutionary setting.

Categories and Subject Descriptors

I.2 [**Artificial Intelligence**]: Multiagent systems

General Terms

Economics, Theory

Keywords

Agent Interaction, Evolution of Cooperation, Emergent behavior

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1. INTRODUCTION

The question of how cooperation emerges among rational, intelligent agents is one which has received considerable amount of attention. When agents interact with one another, they are often faced with two choices - cooperate with each other for mutual benefit, or think about one's own interests, and defect. This abstract interaction model is captured succinctly in the Prisoner's Dilemma (PD) game, which is a two player, one shot, simultaneous move game. Although the Hawk-Dove and Stag Hunt games are also used to model agent cooperation, the PD game is arguably the most often used and well studied.

Table 1 shows the (normalized) payoff matrix for the two player PD game. In this game, each agent can choose one of two actions or strategies – cooperate or defect. The first entry in each cell in the table is the payoff for the row player, and the second entry is that of the column player. For example, when the row player cooperates and the column player defects, then the cell entry is $0, b$, meaning that the row player gets 0, and the column player gets b. When both agents cooperate, each gets a moderate payoff (say 1). However, a defector achieves a higher payoff (say b) against a cooperator, who gets zero payoff. Here b is called the benefit or temptation to defect, and typically $1 < b < 2$. When both agents defect, then both get zero payoff. We observe that if the column player cooperates, the row player is better off defecting, whereas, if the column player defects, then the row player is indifferent as to his strategy. Hence defection is a dominant strategy, and rational agents will be expected to defect always. When two rational players play the PD game, both defect. However, it would ultimately have been better for both agents to have cooperated with each other (mutual cooperation is *Pareto-optimal* with respect to mutual defection) and therein lies the dilemma. Hence it is of interest to study what conditions or protocols of interaction induce agents to cooperate with one another.

	Cooperate	Defect
Cooperate	1,1	0,b
Defect	b,0	0,0

Table 1: Prisoner's Dilemma payoff matrix

In the literature, several settings that sustain cooperation have been proposed and studied. When agents interact repeatedly with one another, and can remember past histories, then agents can retaliate against defectors by refusing to cooperate in future interactions, and this could induce mutual cooperation [3]. However this line of reasoning fails in settings where the same agents may not interact repeatedly. In such settings, reputation mechanisms may be used by the community of agents to track and punish defectors [19]. Such a technique requires unique, constant identities for the defectors, and cannot account for settings in which agents are anonymous, or can freely change their identities, such as happens in interactions over the Internet. Hence we seek interaction models which sustain cooperation in a society of myopic, memoryless agents. We approach this study by looking at the way in which the structure of interaction between the agents affects their strategies.

In a multiagent society, one cannot always expect that all agents interact with one another. For example, spatial structure (agents only interact with other agents in their vicinity) and organizational structure (agents interact only with other agents immediately above or below them in a hierarchy) are two immediate examples in which interactions are restricted between agents. The structure of the interaction of agents with one another is naturally represented by a network, with each node corresponding to an agent, and each edge representing an interaction between a pair of agents. We model agent interaction as an evolutionary PD game played on this network. In this model, every agent is either a cooperator or a defector, and plays the same strategy uniformly with each of its neighbors on the network. We take cooperation to be the incumbent strategy, and defection as the mutant strategy. Starting with a population of cooperators, a small number of agents are switched to defection. The agents then play the game with their neighbors on the network. The fitness of an agent is calculated as the sum of the payoffs that it receives in each game. After one such round the defectors then decide whether to stay with their current strategy, or switch, depending on the fitness of their neighbors. In particular, a defector switches to cooperation when one of its neighbors is a cooperator with higher fitness. We say that a network sustains cooperation if almost all defectors switch to cooperation.

Given this simple model of agent interaction, we ask which networks sustain cooperative behavior, and which do not. In our analysis, we identify some necessary properties that a network should satisfy in order to sustain cooperation, such as small average diameter, densification, and irregularity (Section 4). Real-world networks have been empirically observed to exhibit these properties, and hence can be considered suitable for sustaining cooperation. We also identify some graphs which sustain cooperation, and some which do not (Section 5). In this way, we try to build a complete characterization of networks which sustain cooperation.

There have been many simulation studies on the sustenance of cooperation on networks, but very few of analytical nature. In an analytical study, although the complexity of the model that is studied is necessarily limited in the interest of tractability, the insights obtained are very clear. In fact, it is the analytical approach alone that enables us to obtain a characterization of graphs that sustain cooperation, as illustrated in our necessary conditions above. In contrast, it is arguably difficult, if not impossible to conduct an simulation on a graph class such as “all graphs with large average diameter”.

The rest of the paper is as follows: In Section 2, we describe the network interaction model that we use in our study, and formally define when a network is said to sustain cooperation. In Section 3, we briefly survey the literature relating to the study of sustenance of cooperation under various settings. In Sections 4 and 5, we give proofs for the necessary conditions to sustain cooperation and analyze some specific graphs, respectively. Finally, in Section 6, we summarize our study, and also identify some avenues for future work.

2. THE NETWORK INTERACTION MODEL

2.1 Intuition for the Model

Our definition of networks which sustain cooperation uses a fitness function and imitation rule that follows in spirit the model described by Kearns and Suri [13]. However, we have made some important changes to the model to make it more realistic and suitable to a wider range of applications. We now describe the salient features of our model.

1. **Agent fitness**: The fitness of each agent is defined to be the sum of the payoffs it derives from the PD games played with each of its neighbors in the interaction network. We have defined it to be the sum rather than average (as considered in [13]), because in real world networks, the agents are disparate and have fitnesses based on their centrality and degree of connectedness and normalizing them based on their degree is not natural. The sum of payoffs has been used in earlier works as a measure of fitness of an agent in evolutionary game theory [1, 17].

2. **Incumbents and mutants**: In [13], it is shown that an evolutionarily stable strategy (ESS) in a classical sense (that is, when underlying network of interactions is a complete graph) is also an ESS on graphs (with respect to their fitness model). This result holds under mild restrictions on the graph or how the mutants are chosen. In the context of the PD game, this implies that a population of defectors is resistant to invasion by mutant cooperators. However, we are trying to study whether cooperation (a dominated strategy) can survive mutant attacks (by defectors) given certain network topologies. To this effect we assume the incumbent strategy is cooperation and introduce defectors as mutants.

3. **Imitation dynamics**: In a PD game, for every agent, the best response strategy is to defect (irrespective of the actions of other agents). Hence cooperation cannot be sustained when agents follow best response dynamics. The dynamics considered in our work is not that of best response but that of imitation (which not only applies to humans but also to primitive life forms where rationality cannot be completely justified). Here, if a mutant defector has at least one cooperator neighbor with a higher fitness than it, then it is likely to imitate that strategy and switch over to cooperation. This is a natural behavior and models the fact that every agent tries to imitate other successful agents it interacts with, so as to improve its own payoff [8].

4. **Sustaining cooperation**: Starting with a network of cooperators, a small randomly selected fraction of agents are switched to defection. We say that a network sustains cooperation, if almost all defectors change

their strategy back to cooperation, in accordance with the imitation dynamics described above. The random selection is warranted, as with adversarial placement it is always possible to place mutants in such a way that no mutant switches to cooperation, and the question becomes trivial.

2.2 Notation

Following the game model considered by Santos and Pacheco [20], we work with a normalized PD game matrix, shown in Table 1. The interaction structure among the agents is specified by a graph $G = (V, E)$, where V is the vertex (or node) set, and $E \subseteq V \times V$ is the edge set. The number of vertices (|V|) is denoted as n. We consider graphs which are specified for all large values of n, as we are interested in asymptotic behavior as $n \to \infty$. When two vertices u, v are connected by an edge, we denote the edge as (u, v) (which is equivalent to (v, u), because we work with undirected graphs), and say that $(u, v) \in E$. For each vertex $v \in V$, the neighborhood $N(v)$ is the set of vertices adjacent to v. That is, $N(v) = \{u \in V : (u, v) \in E\}$. Each vertex v corresponds to an agent, and has a fixed strategy $s(v)$, which is either C or D. In future, we will refer to the terms agent and vertex interchangeably. Also, we will refer to each agent, or node, as either a cooperator or a defector, depending on its strategy. Now each agent plays the PD game with each of its neighbors on the graph. Denote by $f(v, u)$, the payoff obtained by agent v playing against agent u. Now the total fitness $f(v)$ of agent v is the sum of its payoffs against each of its neighbors, that is, $f(v) = \sum_{u \in N(v)} f(v, u)$.

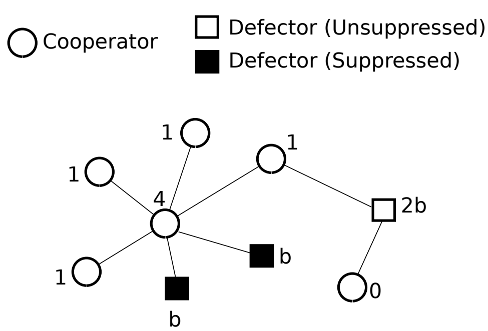

Figure 1: Network Interaction Model

Figure 1 shows an example graph with some cooperators and defectors, along with their fitness. We also see two defectors (shaded) adjacent to a cooperator who has a higher fitness than them. We say that such defectors are *suppressed*. If a defector is not adjacent to any cooperator of higher fitness than it, then we refer to it as *unsuppressed*. We say that an event occurs with high probability, if it occurs with probability $1 - o(1)$ (where $o(1)$ is a term which goes to 0 as $n \to \infty$). Given a graph G, and a constant ϵ^*, $(0 \leq \epsilon^* \leq 1)$ the defector selection process chooses a random subset of vertices D, of size $\epsilon^* n$ as defectors, keeping the other $(1 - \epsilon^*)n$ vertices C as cooperators. (Strictly speaking, the number of cooperators and defectors should be integers. Our analysis can be carried out taking either the floor or ceiling of such quantities, without affecting the results.) A graph G is said to sustain cooperation, if there is a constant ϵ $(0 < \epsilon < 1)$ such that for all values $\epsilon^* \leq \epsilon$, with high probability (with respect to the defector selection process), at most $o(n)$ defectors are unsuppressed. If not, that is, for all values of ϵ, there exist corresponding values of $\epsilon^* \leq \epsilon$, such that the probability that the number of unsuppressed defectors is $\Omega(n)$ is greater than some fixed α, then we say that the graph does not sustain cooperation. The definition of sustenance of cooperation of a graph, can be extended to that of a random graph in a simple manner. In this case, we require that with high probability with respect to selection of a graph from the set of possible graphs (in addition to selection of the defector set), at most $o(n)$ defectors are unsuppressed.

In the above model, we select a random subset of nodes as defectors. One might also consider an adversarial selection. In Section 4.1, we show that with adversarial selection of the defector set, no graph can sustain cooperation, and hence the model becomes uninteresting.

3. RELATED WORK

We now briefly survey the literature which addresses the broad question of how cooperation emerges and is sustained in social interactions, and also identify some key differences between earlier work and ours. These studies can be categorized according to the choice of conditions studied. The most commonly used and popular model of agent interaction is the Prisoner's Dilemma (PD) game, although some studies use the Hawk-Dove (HD, also called Snowdrift or Chicken) game [10, 20]. The network of interaction between the agents can be either static [11, 20] or dynamic [9, 12]. The agents themselves can be myopic and memoryless [11, 20] or strategic and intelligent [3, 12].

Early studies relating to sustenance of cooperation have been in the context of repeated games, where agents play the PD game repeatedly with one another for an infinite number of rounds. In such a setting, the Folk Theorem indicates that mutual cooperation can be sustained as a Nash equilibrium [18]. Axelrod and Hamilton [3] found experimentally that cooperation is sustained when defectors are punished reciprocally with defection in a tit-for-tat fashion. The study was conducted in the form of two computer tournaments where strategic programs competed against one another in a repeated PD game. In their model, each agent (program) plays with every other agent in several rounds, can distinguish each agent's identity, remember the history of actions for each agent that it played, and adopt a different strategy for each interaction. Also the final fitness of an agent is calculated as the sum of the payoffs that it received in each round. A detailed analysis of the sustenance of cooperation in such a setting can be found in [2]. In contrast to this model, agents in our model can adopt only one strategy at a time, which they uniformly exercise in all their interactions, and the payoffs that agents receive are recomputed in each round and are not added up.

Hofmann, Chakraborty, and Sycara [11] carried out a comprehensive simulation study, in which they found that the sustenance of cooperation depends on a number of factors such as network topology, strategy update rule, and initial population of cooperators. In particular, they found that scale-free networks sustain cooperation given almost any update rule. Hanaki, Peterhansl, Dodds, and Watts [9] conducted simulations and found that cooperation can be sustained in a dynamic multiagent network when links between agents are costly and local structure is largely absent. They used a model where each agent imitates the strategy of its most successful neighbor, and breaks/creates links stochastically based on a cost/benefit comparison. Santos and Pacheco [20] carried out simulations of the PD and HD games on various networks and found that cooperation is not sustained on regular graphs and graphs formed by a growth model without preferential attachment, but can be sustained on graphs formed by a growth model with preferential attachment (such graphs were introduced by Barabasi and Albert [4]). Other simulation studies of the sustenance of cooperation are [1, 6, 10, 15, 17, 21]. Also refer to Szabo and Fath [22] for a survey of evolutionary games on graphs.

We observe that the literature in this field largely deals with simulation studies of the sustenance of cooperation on networks. We now turn to the relevant analytical studies in this area. The work which is most similar to ours in terms of the techniques used is that of Kearns and Suri [13], who extended the notion of evolutionarily stable strategies (ESS, [16]) to games played on graphs. An ESS is a strategy that is resistant to invasion by mutant strategies. Kearns and Suri showed analytically that the ESS of games are preserved in their model. In the context of the PD game, this implies that defection is dominant even in the graph setting. While the techniques that we use are similar to the above work, our model is fundamentally different in the computation of fitness, as well as in the restrictions imposed on the graph and placement of defectors. Hence our analysis yields qualitatively different results regarding the sustenance of cooperation, which cannot be obtained in their model. Immorlica, Lucier, and Rogers [12] found that cooperation can be sustained by the formation of social capital. They studied a PD game on a dynamic network where an agent can change its links, but not its strategy. At each round some randomly chosen agents are removed, and replaced by new agents, who choose their strategy based on expected long term fitness. They found that under some parameter settings, cooperators and defectors co-exist in a dynamic self-correcting equilibrium. Our work is different from this, in that we consider myopic, memoryless agents, who cannot compute long-term costs and benefits, but are instead driven by imitation dynamics.

3.1 Our Contributions

In the context of the proposed model, we identify the following necessary conditions for a network to sustain cooperation:

- **Small Average Diameter**: The average diameter of the network should be sub-linear in the number of nodes. Real-world networks have been shown to have an average diameter that grows logarithmically in the number of nodes [23].

- **Densification**: As the number of nodes in the network grows, the average degree of the nodes should increase. In other words, the number of edges should grow super-linearly in the number of nodes [14].

- **Irregularity**: The ratio of maximum degree to minimum degree should be greater than b, the benefit received by defectors. Real-world networks have a power law degree distribution, and hence satisfy this condition [4].

In particular, we analyze the sustainability of cooperation on specific important networks, and classify them accordingly:

- **Sustaining cooperation**: Scale-Free graphs, Hierarchical graphs, Bipartite Random graphs

- **Not sustaining cooperation**: Erdos-Renyi random graphs

4. NECESSARY CONDITIONS FOR SUSTAINING COOPERATION

First, we show that with adversarial selection of the defector set, no graph can sustain cooperation.

4.1 Adversarial Mutant Selection

THEOREM 1. *For any graph G with n nodes, it is always possible to place ϵn defectors and $(1-\epsilon)n$ cooperators in such a way that for every cooperator v_c and defector v_d adjacent to each other, $f(v_d) \geq f(v_c)$.*

PROOF. Let P be a placement of cooperators and defectors on the graph which minimizes the total fitness of all cooperators. We will show that this placement satisfies the condition of the theorem, and we are done.

If not, then there is a cooperator v_c, and a defector v_d adjacent to each other, such that $f(v_c) > f(v_d)$. Let the number of cooperators who are adjacent to v_c but not v_d be k_c and those adjacent to v_d but not v_c be k_d, and let k be the number of cooperators adjacent to both. Then, $f(v_c) = k_c + k; f(v_d) = b \cdot (k_d + k); f(v_c) > f(v_d) \Rightarrow k_c + k > b \cdot (k_d + k) \Rightarrow k_c > k_d$. Now interchange the strategies of v_c and v_d. That is, v_c now becomes a defector, and v_d becomes a cooperator. Let us consider the total change in the cooperator fitness. The fitness of each of the k_c cooperators adjacent to v_c decreases by 1 ($-k_c$); that of the k_d cooperators adjacent to v_d increases by 1 ($+k_d$); and that of the k cooperators adjacent to both does not change. Also, v_c is no longer a cooperator ($-k_c - k$) and v_d is now a cooperator ($+k_d + k$). Hence the change in total cooperator fitness is $2(k_d - k_c)$, which is negative since $k_d < k_c$. Now we have a new placement of cooperators and defectors P', in which the total cooperator fitness is strictly less than that of P, contradicting the minimality of P. $\square$

We now establish two key lemmas that will be used in the proofs.

LEMMA 1. *Let A_1, A_2, ..., A_k be a set of events, where k is a constant. Further, each A_i occurs with high probability Let $A = \bigcap_{i=1}^{k} A_i$. Then, A occurs with high probability.*

PROOF. For each i, since $Pr[A_i] = 1 - o(1)$, $Pr[\overline{A_i}] = o(1)$. Also, $Pr[A] = 1 - Pr[\overline{A}]$, and $\overline{A} = \bigcup_{i=1}^{k}(\overline{A_i})$. By the union bound, $Pr[\bigcup_{i=1}^{k}(\overline{A_i})] \leq \sum Pr[\overline{A_i}] = o(1)$. $\square$

LEMMA 2. *Let a graph G have $\Omega(n)$ vertices of bounded degree. Then G does not sustain cooperation.*

PROOF. Let V_S be the set of vertices of bounded degree, that is, which have degree at most some k. We are given that for some fixed β, $|V_S| \geq \beta n$. (Actually, this statement is true when $n > n_0$, for some fixed n_0, but we are interested in the asymptotic behavior as $n \rightarrow \infty$, so we do not mention this condition.) Set $\epsilon^* = \epsilon$. For each vertex in this set, the probability that it and its neighbors are defectors is at least ϵ^{k+1}, and expected number of such vertices is at least $\beta\epsilon^{k+1}n$, which is linear in n. It is clear that these vertices are unsuppressed, and hence the total number of unsuppressed defectors is at least $\beta\epsilon^{k+1}n$. By the Large Expectation Lemma (refer A.2), we can say that with constant non-zero probability, the number of unsuppressed defectors is greater than $o(n)$. □

We now establish the necessary conditions for any graph to sustain cooperation.

4.2 Small Average Diameter

With a view to keeping our model as general as possible, we do not require the networks under consideration to be connected. That is, the network may consist of disjoint sets of vertices, which are not connected to each other. Each such set of vertices, within which there is a path between any pair of vertices is called a component. We now define the diameter and average diameter of a component G' or a network G. The distance between two nodes in a component is defined as the length of the shortest path between them. The diameter ($diam(G')$) and average diameter ($diam_{avg}(G')$) of a component G' are defined as the maximum and average distance respectively over all pairs of nodes within the component. The diameter ($diam(G)$) and the average diameter $diam_{avg}(G)$ is defined as the maximum over all components of the diameter and average diameter respectively.

We find that in order to sustain cooperation, the average diameter of the network should be $o(n)$, that is, the average diameter should grow sub-linearly in the number of nodes. In other words, the network cannot have a large average diameter of $\Omega(n)$.

THEOREM 2. *Let G be a graph of large average diameter, that is, $diam_{avg}(G) = \Omega(n)$. Then G does not sustain cooperation.*

PROOF. From the definition for average diameter, we know that some component of G, say G', has average diameter $diam_{avg}(G') = \Omega(n)$. It is easy to see that this implies $diam(G') = \Omega(n)$ and also that G' has $\Omega(n)$ vertices. We now show that some $\Omega(n)$ vertices in G' have bounded degree. This along with Lemma 2 establishes the theorem.

Suppose not, that is, at most $o(n)$ vertices in G' have bounded degree. Call this set V_S. All the other vertices in G' have degree $\omega(1)$. Call this set V_B. Recursively apply the following procedure:

1. Let $i = 0$; $V' = V_B$
2. Find $v \in V' : N(v) \bigcap (S_0 \biguplus S_1 .... \biguplus S_{i-1}) = \phi$ (Stop if no such vertex exists).
3. Let $N[v] = v \cup N(v)$
4. Assign $S_i = N[v]$; $i = i + 1$
5. Assign $V' = V' - N[v]$

Let the value of i at the end of the iterations be k. Since each S_i has $\omega(1)$ vertices, the number k of stars S_i formed is $o(n)$. Notice that each vertex remaining in V' at the end is at distance one to some star, that is, it has a neighbor in some star S_i. For each vertex v remaining in V', find some i such that $S_i \cap N(v) \neq \phi$, and add v to S_i. Now the diameter of each S_i is at most 4.

The component G' is now decomposed into two parts - $o(n)$ stars $(S_0, ..., S_{k-1})$, and some subset V'_S of V_S. Let us look at the shortest path between any two vertices in G'. This path passes through each S_i at most constant times, and through each of the V'_S at most once. Hence the diameter is $o(n)$, contradicting the statement above that $diam(G') = \Omega(n)$. □

4.3 Densification

A graph is said to densify if the number of edges in the graph asymptotically grows faster than n. That is, $|E| = \omega(n)$. Put in another way, these are graphs whose average degree increases with n. This rules out all sparse graphs (paths, cycles, trees, planar graphs, etc.), that is, graphs which do not densify over time.

THEOREM 3. *Let G be a sparse graph, that is, $|E| = O(n)$. Then G does not sustain cooperation.*

PROOF. We will show that some $\Omega(n)$ vertices have bounded degree, which along with Lemma 2, establishes the result.

We are given that for some fixed β, $|E| \leq \beta n$. Since the total degree is at most $2\beta n$, there can be at most $n/2$ vertices of degree greater than 4β, and hence at least $n/2$ vertices of degree bounded by 4β, which is a constant. □

4.4 Irregularity

The degree $d(v)$ of a vertex v is the size of its neighborhood $N(v)$. The maximum degree Δ and minimum degree δ of a graph G are defined as the maximum and minimum respectively, over the degrees of all vertices of G. For a graph G to sustain cooperation, the ratio of the maximum degree to the minimum degree should be at least b ($\Delta/\delta \geq b$). This means that the graph should be irregular to some extent, ruling out all near-regular graphs.

THEOREM 4. *Let G be a Near-Regular graph, that is, $\Delta/\delta < b$. Then G does not sustain cooperation.*

PROOF. Let $\tau := \Delta/\delta < b$. If $\delta = O(1)$, that is, a constant, then Δ is also a constant, implying that all nodes have bounded degree. By Lemma 2, the graph does not sustain cooperation.

Now consider $\delta = \omega(1)$. We will show that with fixed non-zero probability, there is a linear-sized set of defectors, in which each defector has a fitness higher than that of any other cooperator. No defector in this set is suppressed, and the result follows.

The maximum degree of any vertex is Δ, and hence the maximum fitness of any cooperator is Δ. This implies that if a defector has more than $\frac{\Delta}{b}$ cooperator neighbors, then it is unsuppressed. We call such a node *bad*, and the other nodes *good*. Let us now give an upper bound for the probability of a node being good. This happens if the node is a cooperator (probability $1 - \epsilon^*$), or if the node has less than $\frac{\Delta}{b}$ cooperator neighbors. Using the Chernoff bound, this latter probability is not more than $e^{-\frac{\lambda^2}{3}\mu} =: \beta$, where μ is the expected number of cooperator neighbors of a vertex, which is at least $\delta(1-\epsilon^*)$, and $\lambda = (1 - \frac{\tau}{(1-\epsilon^*)\cdot b})$ is the deviation from the mean on the lower side (which is strictly positive, as ϵ^* can be set to a value less than $1 - \frac{\tau}{b}$). Using the Union bound, the probability of a node being *good* is not more than $1 - \epsilon^* + \beta$, and hence the probability of a node being *bad* is at least $\epsilon^* - \beta =: \gamma$, which is strictly positive (as $\delta = \omega(1)$, β is arbitrarily close to zero).

The expected number of *bad* nodes is at least γn. The maximum number of bad nodes is $\epsilon^* n$, and hence by the Large Expectation Lemma (refer A.2), the size of this set is linear with non-zero probability. □

5. SUSTAINABILITY OF COOPERATION ON SPECIFIC GRAPHS

5.1 Erdos-Renyi Random Graphs

We now analyze the $G(n,p)$ model of Erdos-Renyi [7]. In this model, a graph on n nodes is taken, and the edge between every pair of vertices is included in the graph independently with probability $p(n) = p$. In such a graph, the degree of each vertex is roughly close to the expected value (np). Hence this graph behaves like a regular graph, and one would expect that does not sustain cooperation. This intuition is supported by the proofs for specific ranges of the parameter p ($p = O\left(\frac{1}{n}\right)$ and $p = \omega\left(\frac{log(n)}{n}\right)$). We expect the result to hold similarly for other values of p as well.

THEOREM 5. *Let $G(n,p)$ be an Erdos-Renyi random graph, where $\forall e \in V \times V, Pr[e \in E] = p$, and $p = O\left(\frac{1}{n}\right)$ or $p = \omega\left(\frac{log(n)}{n}\right)$. $G(n,p)$ does not sustain cooperation.*

PROOF. When $p = O\left(\frac{1}{n}\right)$, with high probability the number of edges is linear in n. That is, $|E| = O(n)$. Hence $G(n,p)$ does not densify, and by Theorem 3, does not sustain cooperation.

When $p = \omega\left(\frac{log(n)}{n}\right)$, we show that the graph is near-regular, and hence by Theorem 4, does not sustain cooperation. The expected degree of each vertex is $\mu := E[d(v)] = (n-1)p = \omega(log(n))$. By the Chernoff bound, the probability that the degree of a node is not within $(1 \pm \lambda)\mu$ is at most $2e^{-\frac{\lambda^2}{3}\mu}$, and the expected number of such nodes is $n \cdot 2e^{-\frac{\lambda^2}{3}\mu}$, which goes to 0. Hence all nodes have degrees within $(1 \pm \lambda)$ of the expected value. Setting λ such that $\frac{1+\lambda}{1-\lambda} < b$ makes the graph near-regular, and we have the result. □

5.2 Bipartite Random Graphs

These graphs are bipartite graphs, that is, graphs in which the vertex set can be partitioned into two sets L and R, such that there are no edges within each partition. Also the size of the partition L is given by a function $f(n)$, which asymptotically grows faster than $log(n)$, but slower than n. Every edge between one vertex in L and one in R is included in the graph independently with probability $p(n)$ (which is at least $\frac{4log(n)}{f(n)}$).

We thus have a family of random graphs parametrized by the values of $f(n)$ and $p(n)$. We observe that the expected degrees of vertices in partition L is linear in n, whereas in partition R is sub-linear in n, because of which, cooperators in L can suppress defectors in R, and intuitively these graphs should sustain cooperation. This family of graphs is useful in that allows us to construct networks which sustain cooperation having any given edge density which is super-linear in $n * log(n)$ and sub-linear in n^2, that is $|E| = \omega(n * log(n)), o(n^2)$. This is done by setting $f(n) = \frac{|E|}{n}$, and $p(n)$ appropriately.

THEOREM 6. *Let G be a bipartite graph with $V = L \uplus R$ (L and R are the two partitions of the vertices), $|L| = f(n) = \omega(log(n)), o(n)$, and $\forall e \in L \times R, Pr[e \in E] = p(n) = p$, where $p(n) \geq \frac{4log(n)}{f(n)}$. G sustains cooperation.*

PROOF. We will show that all defectors in R are suppressed. Hence the number of unsuppressed defectors is only that in L, which is at most $f(n) = o(n)$, and the theorem is proved.

Let $0 < \epsilon < 1$ and let ϵ^* take any value $\leq \epsilon$; define $\epsilon' := 1 - \epsilon^*$. The number of cooperators in partition L has an expected value of $\epsilon' f(n)$, and by the Chernoff bound, is at least $\frac{1}{2}\epsilon' f(n)$ with high probability. Call this set V_C.

There are a total of $\epsilon' n$ cooperators in the graph, and hence at least $\epsilon' n - f(n)$ cooperators in R. Since $f(n) = o(n)$, this number is at least $\frac{1}{2}\epsilon' n$. Since a cooperator in V_C is adjacent to each cooperator in R independently with probability p, the expected fitness of a cooperator in V_C is at least $\frac{1}{2}\epsilon' np$. By the Chernoff bound, the probability that the fitness is less than half of this expected value is not more than $e^{-\frac{1}{24}\epsilon' np}$. The expected number of such vertices is not more than $\frac{1}{2}\epsilon' n * e^{-\frac{1}{24}\epsilon' np} = \frac{1}{2}\epsilon' e^{log(n) - \frac{1}{24}\epsilon' np}$, which goes to 0, as $p \geq \frac{48log(n)}{n}$. Hence the fitness of each cooperator in V_C is at least $\frac{1}{4}\epsilon' np$, with high probability.

Using a similar analysis, we can say that the fitness of each defector $d \in R$ is at most $\frac{3}{2}b\epsilon' pf(n)$, with high probability.

From the above two arguments, we can say that any defector in R who is adjacent to any one cooperator in V_C is suppressed.

The probability that a defector in R is not connected to any cooperator in V_C is at most $(1-p)^{\frac{1}{2}\epsilon' f(n)}$, which is at most $e^{-\frac{1}{2}\epsilon' f(n)p}$. The expected number of such defectors is at most $n * e^{-\frac{1}{2}\epsilon' f(n)p}$, which goes to 0 since $p \geq \frac{4log(n)}{f(n)}$. □

5.3 Hierarchical Graph

This graph is built by starting with a complete binary tree, and connecting each node to all of its ancestors (and descendants). The set of nodes of a given degree correspond to a *level* in the hierarchy. Nodes which are higher up in the hierarchy are high degree nodes, and those lower in the hierarchy are of low degree. Hence intuitively cooperators in higher levels can suppress defectors in lower levels, and cooperation can be sustained.

THEOREM 7. *Let G be a complete binary tree where each node is connected to all its descendants. Formally, let Σ^* be the set of all strings over alphabet $\Sigma = \{0,1\}$. For each such $s \in \Sigma^*$, let $|s|$ denote the length of s, and let prefix be a relation on Σ^*, such that prefix(u,v) is true, when the string u is a prefix of the string v. Take $V = \{s \in \Sigma^* : |s| \leq log(n+1) - 1\}$, and $E = \{(u,v) : prefix(u,v) \text{ or } prefix(v,u)\}$. G sustains cooperation.*

PROOF. Let us call the length of the string that represents a vertex v, as the *level* of v; and say that the level of vertex v is *above*, or *higher* than that of vertex u, if $|v| < |u|$ (and also that the level of vertex $|u|$ is *below* or *lower* than that of $|v|$). Let $0 < \epsilon < 1/2$ and let ϵ^* take any value $\leq \epsilon$, and define $\epsilon' := 1 - \epsilon^*$.

First we will show that every vertex at level $h_1 + 1$, where $h_1 = \frac{1}{3}log(n)$, has an ancestor who is a cooperator (and hence every vertex below level h_1 (that is, $|v| > h_1$) has an ancestor of level higher than h_1+1 (that is, $|v| \leq h_1$), who is a cooperator). Every vertex at level h_1+1 has h_1 ancestors, and the probability that all of them are defectors is roughly $(\epsilon^*)^{h_1}$. Hence the expected number of such vertices is $2^{h_1+1} \cdot (\epsilon^*)^{h_1}$, which goes to 0 as $\epsilon^* \leq \epsilon < 1/2$.

Now, we will show that each vertex of level h_1 (and hence each vertex of level higher than h_1+1), has at least $\frac{1}{4}\epsilon' n^{2/3}$ cooperator neighbors. For each vertex at level h_1, the expected number of cooperator neighbors is at least $\frac{1}{2}\epsilon' n^{2/3}$, and by the Chernoff bound, the probability that this number is less than half the expected value is not more than $e^{-\frac{1}{12}\epsilon' n^{2/3}}$. The expected number of such vertices is not more than $n^{2/3} * e^{-\frac{1}{12}\epsilon' n^{2/3}}$, which goes to zero.

It is clear that each defector of level lower than $h_2 = \frac{2}{3}log(n)$ has fitness f_d at most $b \cdot |N(d)| \leq b \cdot (n^{1/3} + log(n))$. By the above two arguments, each such defector has an ancestor of level higher than $h_1 + 1$ who is a cooperator, who has fitness f_c at least $\frac{1}{4}\epsilon' n^{2/3}$. Clearly $f_c > f_d$. Hence each such defector is suppressed. The number of unsuppressed defectors is at most that in levels above $h_2 + 1$, which is $O(n^{2/3}) = o(n)$. □

5.4 Scale-Free Graphs

In order to study the sustainability of cooperator on scale-free networks, we consider the random scale-free graph described in [5]. In this model, the vertices are labeled from 1 through n, and the edge between two vertices is included in the graph with some probability, which is defined as a function of the vertex labels. In our model, $Pr[(i, j) \in E] = (ij)^{-\frac{1}{2}+\kappa}$, where $0 < \kappa < \frac{1}{2}$. The expected degree of node i is given by $E[d(i)] = \frac{2}{1+2\kappa} n^{(\frac{1}{2}+\kappa)} i^{(-\frac{1}{2}+\kappa)}$. Observe that the lesser the label i of a node, the higher will be its degree. Cooperator nodes with lower labels can suppress defector nodes with higher labels, thereby sustaining cooperation.

Theorem 8. *Let graph G have vertex set $|V| = \{v_i : 1 \leq i \leq n\}$, and $\forall i, j \in V, Pr[(i, j) \in E] = (ij)^{-\frac{1}{2}+\kappa}$, where $0 < \kappa < \frac{1}{2}$. G sustains cooperation.*

Proof. Let $0 < \epsilon < 1$ and let ϵ^* take any value $\leq \epsilon$; define $\epsilon' := 1 - \epsilon^*$. Consider α, β, such that $0 < \alpha < \beta < 1$. Let $V_1 = v_i : 1 < i < n^\alpha$; $V_2 = v_i : n^\alpha < i < n^\beta$; and $V_3 = v_i : n^\beta < i < n$. By the Chernoff bound, with high probability there are at least $\frac{1}{2}\epsilon' n^\alpha$ cooperators in V_1. Call this set V_C.

There are at most n^α cooperators in V_1, and hence at least $\frac{n}{2}$ cooperators in $V_2 \uplus V_3$. Call this set V_C'. The expected fitness of any cooperator in V_C – call it f_c – is at least $\sum_{j \in V_C'} \epsilon'(n^\alpha j)^{-\frac{1}{2}+\kappa}$, which is at least $\epsilon' n^{\alpha(-\frac{1}{2}+\kappa)} \sum_{j=n/2}^{n} (j)^{-\frac{1}{2}+\kappa}$ $= \epsilon' n^{\alpha(-\frac{1}{2}+\kappa)}(S(n) - S(n/2))$, where $S(n) = \frac{2}{1+2\kappa} n^{\frac{1}{2}+\kappa}$. Simplifying, we get $f_c \geq 2\gamma n^{\alpha(-\frac{1}{2}+\kappa)+\frac{1}{2}+\kappa}$, where γ is a constant. By the Chernoff bound, we can show that with high probability all cooperators in V_C have fitness higher than $\gamma n^{\alpha(-\frac{1}{2}+\kappa)+\frac{1}{2}+\kappa}$.

Similarly, it can be shown that with high probability each defector in V_3 has fitness at most $\gamma' n^{\beta(-\frac{1}{2}+\kappa)+\frac{1}{2}+\kappa}$. By the above two arguments, with high probability any defector in V_3 who is adjacent to any cooperator in V_C is suppressed (as $\alpha < \beta$).

The probability that a defector in V_3 is not connected to any cooperator in V_C is at most $\prod_{n^\alpha}^{(1-\epsilon')n^\alpha} (1 - (nj)^{-\frac{1}{2}+\kappa})$, which is at most $\gamma e^{-n^{2\kappa}}$, where γ is some constant. The expected number of such defectors is at most $n \cdot \gamma e^{-n^{2\kappa}}$ which goes to zero. Hence with high probability all defectors in V_3 are suppressed, and the number of unsuppressed defectors is at most the size of $|V_1 \uplus V_2|$, which is $o(n)$. □

6. CONCLUSIONS AND FUTURE WORK

We have studied the Evolutionary Prisoner's Dilemma on graphs as a model of cooperation. In particular, we identify the role played by the network topology in sustaining cooperation in a multiagent society. We have shown analytically that for a network to sustain cooperation, it must exhibit properties such as small average diameter, densification, and irregularity. Real-world networks exhibit these properties, and hence could be suitable for sustaining cooperation. Also, we have shown that a family of Scale-Free graphs, a Hierarchy, as well as a family of Bipartite Random graphs sustain cooperation, whereas Erdos-Renyi random graphs do not.

Further exploration along these lines can be carried out to determine which graphs sustain cooperation and which do not, building towards a complete characterization. In our interaction model, we have considered one particular imitation rule (a mutant copies the incumbent strategy if it has an incumbent neighbor with higher fitness) and one particular fitness function (sum of the payoffs received in games played with all neighbors) in the context of the PD game. A similar analysis can be carried out in the context of other strategy update rules and fitness functions, as well as other models of cooperation, such as the Hawk-Dove and Stag-Hunt games.

In general, we feel the kind of analytical treatment carried out in this work yields interesting insights into the factors influencing the sustenance of cooperation, complements the simulation work in the literature, and warrants further studies along similar lines.

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APPENDIX

A. COMMONLY USED FORMULAE

A.1 Chernoff Bound

Let $(X_1, X_2, ..., X_n)$ be a set of independent random variables, each of which takes value 0 (with probability $1 - p_i$) and value 1 (with probability p_i). Also, let $X = \sum_{i=1}^{n} X_i$, with $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} p_i$, and $0 \le \lambda \le 1$. Then,

$$\mathbf{Pr}[X \ge (1+\lambda)\mu] \le e^{-\frac{\lambda^2}{3}\mu},$$

$$\mathbf{Pr}[X \le (1-\lambda)\mu] \le e^{-\frac{\lambda^2}{3}\mu}$$

.

A.2 Large Expectation Lemma

THEOREM 9. *Let $X(n)$ be a non-negative discrete (integer) random variable with density function $p(x)$ and maximum value $M(n)$, with $\mathbf{E}[X(n)] \ge \alpha M(n)$, where $0 \le \alpha \le 1$. Let $f(n) = o(M(n))$. Then $\mathbf{Pr}[X \ge f(n)] \ge \alpha/2$.*

PROOF.

$$\mathbf{E}[X(n)] = \sum_{x=0}^{M(n)} xp(x) = \sum_{x=0}^{f(n)-1} xp(x) + \sum_{x=f(n)}^{M(n)} xp(x)$$

$$\sum_{x=f(n)}^{M(n)} M(n)p(x) \ge \sum_{x=f(n)}^{M(n)} xp(x)$$

$$\Rightarrow \mathbf{Pr}[X \ge f(n)] = \sum_{x=f(n)}^{M(n)} p(x) \ge \frac{1}{M(n)} \sum_{x=f(n)}^{M(n)} xp(x)$$

$$\ge \frac{1}{M(n)} \left(\mathbf{E}[X(n)] - \sum_{x=0}^{f(n)-1} xp(x) \right)$$

$$\ge \frac{1}{M(n)} \left(\mathbf{E}[X(n)] - \sum_{x=0}^{f(n)} f(n)p(x) \right)$$

$$\ge \frac{\mathbf{E}[X(n)]}{M(n)} - \frac{f(n)}{M(n)} \sum_{x=0}^{f(n)} p(x)$$

$$\ge \alpha - \frac{f(n)}{M(n)}$$

$$\text{For large enough n, } \frac{f(n)}{M(n)} \le \frac{\alpha}{2}$$

$$\Rightarrow \mathbf{Pr}[X \ge f(n)] \ge \frac{\alpha}{2}$$

□