Complexity and Approximability of Social Welfare Optimization in Multiagent Resource Allocation

(Extended Abstract)

Nhan-Tam Nguyen, Trung Thanh Nguyen, Magnus Roos, and Jörg Rothe
Institut für Informatik, Heinrich-Heine-Universität Düsseldorf, 40225 Düsseldorf, Germany
Nhan-Tam.Nguyen@uni-duesseldorf.de, {thanh, roos, rothe}@cs.uni-duesseldorf.de

ABSTRACT

An important task in multiagent resource allocation, which provides mechanisms to allocate bundles of (indivisible and nonshareable) resources to agents, is to maximize social welfare. We study the computational complexity of exact social welfare optimization by the Nash product, which can be seen as a sensible compromise between the well-known notions of utilitarian and egalitarian social welfare. When utilitiy functions are represented in the bundle or the k-additive form, for $k \geq 3$, we prove that the corresponding computational problems are DP-complete (where DP denotes the second level of the boolean hierarchy over NP), thus confirming two conjectures raised by Roos and Rothe [10]. We also study the approximability of social welfare optimization problems.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences—Economics

General Terms

Economics, Theory

Keywords

Multiagent resource allocation, social welfare optimization, computational complexity, auction and mechanism design

1. INTRODUCTION

In multiagent resource allocation (see, e.g., the survey by Chevaleyre et al. [2]) agents have preferences over bundles of resources. We consider preference representation by utility functions and assume that resources are indivisible and nonshareable. Taking the preferences of agents into account, the task is to allocate bundles of resources to agents. By aggregating the agents' utilities we arrive at the notion of social welfare with which we can assess the quality of allocations from the viewpoint of a global system designer.

One approach is the prominent *utilitarian social welfare*, which is the sum of the agents' utilities and which measures the average benefit every agent achieves. Utilitarian social welfare, however, lacks "fairness" because the utilities that agents realize in a given allocation can differ greatly. Interpreting the utilities as bids or valuations in a combinatorial auction, utilitarian social welfare corresponds to an auctioneer's revenue.

Egalitarian social welfare, in contrast, looks at the agent that is worst off and seeks to improves this agent's utility. While this concept provides some measure of fairness when the minimum needs of all agents are to be satisfied, it does have some disadvantages; for example, it is not strictly monotonic: Raising the utility of an agent who is not worst off does not increase egalitarian social welfare.

The *Nash product*, the product of the agents' utilities, can be seen as a compromise between these two approaches. On the one hand, it has the monotonicity property of utilitarian social welfare because an increase in any agent's utility leads to an increase of the Nash product (provided all agents have positive utility). On the other hand, the Nash product increases as well when reducing inequitableness among agents by redistributing utilities, thereby providing a measure of fairness. Looking at the ordering that is induced by the allocations, the *social welfare ordering*, Moulin [5] presents further beneficial properties of the Nash product. For example, the Nash product is uniquely characterized by independence of individual scale of utilities, i.e., even if different "currencies" are used to measure the agents' utilities, the social welfare ordering remains unaffected.

Having a measure for the quality of allocations, it is a natural task to optimize social welfare, and to ask for the computational complexity of this task.

2. PRELIMINARIES

Multiagent Resource Allocation

Let $A = \{a_1, \ldots, a_n\}$ be the set of *agents*, $R = \{r_1, \ldots, r_m\}$ the set of *resources* (which each are assumed to be indivisible and nonshareable), and let $U = \{u_1, \ldots, u_n\}$ be the set of the agents' *utility functions*. The mapping $u_i : 2^R \to \mathbb{F}$ is agent a_i's utility function, where 2^R denotes the power set of R and $\mathbb{F}$ is a numerical set (such as the set $\mathbb{N}$ of nonnegative integers, the set $\mathbb{Q}$ of rational numbers, and the set $\mathbb{Q}^+$ of nonnegative rational numbers). Such a triple (A, R, U) is called a *multiagent resource allocation setting* (a *MARA setting*, for short). An *allocation* for a given MARA setting (A, R, U) is a mapping $X : A \to 2^R$ with $\bigcup_{a_i \in A} X(a_i) = R$ and $X(a_i) \cap X(a_j) = \emptyset$ for any two distinct agents a_i and a_j. The set of all allocations for a MARA setting (A, R, U) is denoted by $\Pi_{A,R}$ and has cardinality n^m. We use the shorthand $u_i(X)$ to denote the utility $u_i(X(a_i))$ agent a_i can realize in allocation X. We consider the following representation forms for utility functions:

1. The *bundle form*: A utility function u is represented by a list of pairs $(R', u(R'))$ for any bundle $R' \subseteq R$, where pairs with $u(R') = 0$ are dropped.

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2. The *k-additive form* (Chevaleyre et al. [3] and Conitzer et al. [4]), for some fixed positive integer k: A utility function $u : 2^R \to \mathbb{F}$ is in k-additive form if there are coefficients $\alpha_T \in \mathbb{F}$ for each bundle $T \subseteq R$ with $\|T\| \leq k$ such that for any bundle $R' \subseteq R$ the following holds:

$$u(R') = \sum_{T \subseteq R', \|T\| \leq k} \alpha_T$$

DEFINITION 1. *For a MARA setting* (A,R,U) *and an allocation* $X \in \Pi_{A,R}$*, define*

1. *the* egalitarian social welfare of X *as* $sw_e(X) = \min_{a_i \in A}\{u_i(X)\}$*;*
2. *the* Nash product of X *as* $sw_N(X) = \prod_{a_i \in A} u_i(X)$.
3. *As an additional notation, for* $S \in \{e, N\}$*, denote the* maximum egalitarian/Nash product social welfare of a MARA setting $M = (A,R,U)$ *(or of a problem instance that contains a MARA setting* M*) by*

$$\max_S(M) = \max\{sw_S(X) \mid X \in \Pi_{A,R}\}.$$

For $\mathbb{F} \in \{\mathbb{N}, \mathbb{Q}^+, \mathbb{Q}\}$ and $\texttt{form} \in \{\texttt{bundle}, k\texttt{-additive}\}$, define:

$\mathbb{F}$-NASH PRODUCT SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$	
Given:	A MARA setting $M = (A,R,U)$, where `form` indicates how every $u_i : 2^R \to \mathbb{F}$ in U is represented, and $t \in \mathbb{F}$.
Question:	Is there an allocation $X \in \Pi_{A,R}$ such that $sw_N(X) \geq t$?

which we abbreviate by $\mathbb{F}$-NPSWO$_{\texttt{form}}$. The exact version of this problem is denoted by $\mathbb{F}$-EXACT NASH PRODUCT SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$ (or $\mathbb{F}$-XNPSWO$_{\texttt{form}}$, for short) and asks, given a MARA setting $M = (A,R,U)$ and $t \in \mathbb{F}$, whether $\max_N(M) = t$. The corresponding problems for utilitarian and egalitarian social welfare can be defined analogously and have been studied, e.g., by Chevaleyre et al. [2] and Roos and Rothe [10].

As the goal is to find a feasible allocation that maximizes social welfare, we also consider the corresponding maximization problems.

$\mathbb{F}$-MAXIMUM EGALITARIAN SOCIAL WELFARE$_{\texttt{form}}$	
Input:	A MARA setting $M = (A,R,U)$, where `form` indicates how every $u_i : 2^R \to \mathbb{F}$ in U is represented.
Output:	$\max_e(M)$.

As a shorthand, write $\mathbb{F}$-MAX-ESW$_{\texttt{form}}$. Based on sw_N, define $\mathbb{F}$-MAXIMUM NASH PRODUCT SOCIAL WELFARE$_{\texttt{form}}$ (or $\mathbb{F}$-MAX-NPSW$_{\texttt{form}}$) accordingly.

Complexity Theory and Theory of Approximation

We assume that the reader is familiar with the basic notions of computational complexity theory (see, e.g., the textbooks by Papadimitriou [8] and Rothe [11]).

Papadimitriou and Yannakakis [9] introduced the complexity class $\text{DP} = \{L_1 - L_2 \mid L_1, L_2 \in \text{NP}\}$, which contains the differences of any two NP-problems.

DEFINITION 2. *An* α-approximation algorithm *for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of* α *of the value of an optimal solution.*

DEFINITION 3. *A maximization problem* Π *has a* polynomial-time approximation scheme *(*PTAS*) if for every* ε, $0 < \varepsilon < 1$*, there exists an* ε*-approximation algorithm for* Π.

3. RESULTS

We use a sufficient condition for DP-hardness by Chang and Kadin [1] to obtain the following complexity results:

THEOREM 4. $\mathbb{Q}^+$-XNPSWO$_{\texttt{bundle}}$ *is* DP*-complete.*

THEOREM 5. *For each* $k \geq 3$*,* $\mathbb{Q}^+$-XNPSWO$_{k\texttt{-additive}}$ *is* DP*-complete.*

Turning to approximability, we note that a reduction mentioned in [10] and attributed to a reviewer of that paper provides these kinds of inapproximability results.

PROPOSITION 6. *The problems* $\mathbb{Q}$-MAX-ESW$_{\texttt{bundle}}$ *and* $\mathbb{Q}$-MAX-NPSW$_{\texttt{bundle}}$ *cannot be approximated within any factor in polynomial time, unless* P = NP*. This result holds even when the utilities are restricted to the domain* $\{0,1\}$.

THEOREM 7. $\mathbb{Q}^+$-MAX-NPSW$_{\texttt{1-additive}}$ *can be solved exactly in polynomial time when the number of agents and resources are the same and the empty bundle has always utility zero.*

THEOREM 8. *There is a PTAS for* $\mathbb{Q}^+$-MAX-NPSW$_{\texttt{1-additive}}$ *when restricted to only two agents having the same utility function* u *with* $u(\emptyset) = 0$.

More details can be found in [6, 7].

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