Pareto Efficient Strategy-proof School Choice Mechanism with Minimum Quotas and Initial Endowments*

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ABSTRACT

This paper develops a strategy-proof and Pareto efficient mechanism for a school choice program called Top Trading Cycles among Representatives with Supplementary Seats (TTCR-SS). We consider a setting where minimum quotas are imposed for each school, i.e., a school is required to be assigned at least a certain number of students to operate, and the obtained matching must respect initial endowments, i.e., each student must be assigned to a school that is at least as good as her initial endowment school. Although minimum quotas are relevant in school choice programs and strategy-proofness is important to many policymakers, few existing mechanisms achieve both of them simultaneously. Furthermore, existing mechanisms require that all students consider all schools acceptable to obtain a feasible matching that respects minimum quotas and cannot guarantee Pareto efficiency. TTCR-SS is based on Top Trading Cycles (TTC) mechanism, while it is significantly extended to handle the supplementary seats of schools while respecting minimum quotas. Our simulation results show TTCR-SS is significantly better than an existing TTC-based mechanism in terms of students' welfare.

Keywords

Mechanism design; School choice; Resource allocation; Top trading cycles

1. INTRODUCTION

Traditionally, a student is assigned to a public school based on where she lives. School choice programs are implemented to give students/parents an opportunity to choose public schools. A seminal work [3] introduces the idea of using a mechanism design approach for this issue by formalizing it as a problem of allocating indivisible objects with multiple supplies (e.g., schools with seats) to agents (e.g., students).

In this paper, we consider the case where minimum quotas are imposed, i.e., a school is required to be assigned at least a certain number of students to operate. Also, we assume each student has a default school (that she would have attended without the school choice program), which we call her **initial endowment**. We also assume the initial endowments satisfy all minimum quotas and require that a student be allocated to a school that is at least as good as her initial endowment school. This requirement is natural since a student would not want to attend a public school located remotely away from her residence unless it offers some very appealing characteristics.

Although minimum quotas are relevant in school choice programs and strategy-proofness (i.e., no student ever has any incentive to misreport her preference, regardless of other students' reports) is important to many policymakers, there is a lack of strategy-proof mechanisms that consider them. A notable exception [10] develops strategy-proof mechanisms based on the Deferred Acceptance mechanism [11] that can handle minimum quotas. However, there are two limitations for applying their mechanisms in our setting. First, to guarantee that their mechanisms obtain feasible matchings (which respect minimum quotas), students must consider all of the schools **acceptable**. This requirement is unrealistic in our setting. The school choice program is intended to provide more choices to students/parents. However, with this requirement, students/parents are not allowed to declare that some schools are unacceptable. Second, their mechanisms cannot guarantee Pareto efficiency, which is a standard efficiency criterion in economics.

Our newly developed mechanism is based on Top Trading Cycles (TTC) mechanism [26], which is a standard way to improve students' welfare with initial endowments. In our setting, a school may have supplementary/empty seats, i.e., it can accept more students than the initial endowment students. By allocating supplementary seats, the welfare of students can be improved. To the best of our knowledge, there exists no TTC-based mechanism that can handle both initial endowments and supplementary seats while respecting minimum quotas.

Our mechanism is general enough to be applied to any allocation problem of indivisible objects with multiple supplies, where each agent has her initial endowment and minimum quotas are imposed on the number of supplies for each object. One possible application domain of our mechanism is reallocation in a student-laboratory assignment problem. In many universities in Japan, an undergraduate engineering student must be assigned to a laboratory to conduct a project. However, it is difficult for a student to choose an appropriate laboratory since her knowledge is limited. One

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possible remedy is to apply the following three-step procedure: (i) students are assigned to laboratories using some mechanism (e.g., [10]), (ii) students experience a certain trial period, and (iii) each student has a chance to apply to another laboratory if her interest changes or her current laboratory fails to meet her expectations. Our new mechanism can be used in Step (iii). It is natural to require that no student is reallocated to a laboratory that is worse than her current assignment. Also, it is natural to assume the current allocation is feasible.

The rest of this paper is organized as follows. We show a more detailed literature review in the rest of this section. Then we introduce a formal model of our problem setting (Section 2). Next we show a simple mechanism based on TTC (Section 3), introduce our newly developed mechanism (Section 4), and show its theoretical properties (Section 5). Finally, we evaluate our mechanism by computer simulation (Section 6).

1.1 Related literature

The problem of allocating objects to agents, when objects are initially owned by agents who have strict preferences over objects, is formulated as a housing market problem [26]. In the paper, they introduce TTC due to David Gale, to show that the core is nonempty. Roth and Postlewaite [25] further show that the core is a singleton for strict preferences. For the incentive property, Roth [23] shows that TTC is strategy-proof. Ma [19] shows that a trading mechanism is individual rational, Pareto efficient and strategy-proof if and only if it is TTC. Later, TTC is generalized to the hierarchical exchange mechanism [21] and to trading cycles mechanism [22]. Abdulkadiroğlu and Sönmez [2] consider a setting that resembles ours. In their setting, some houses are initially owned by tenants while others are not. They modify TTC to this setting. The differences between [2] and our work are that we consider the setting where multiple copies of objects (seats of schools) exist and minimum quotas are imposed. Erdil and Ergin [9] consider a setting in which priorities are not strict. They also modify TTC to this setting and develop the stable improvement cycles mechanism. This mechanism is stable, but not strategy-proof.

School choice programs are identified as an important application domain of TTC [3]. This work formulates a school choice problem, in which the seats of schools are allocated to students. Here, a school can have multiple seats, and each school can have an idiosyncratic priority among students. They also introduce a modified version of the original TTC that is specific to a school choice problem, and the mechanism is Pareto efficient and strategy-proof. Since then, TTC in the setting of a school choice problem has drawn independent research interest, and there are many directions of research. One direction of research is to design the priority structure of students for a given mechanism [13, 15]. Another direction of research is using an axiomatic approach to characterize TTC [1, 6, 20].

In our paper, we consider a new kind of institutional constraint: minimum quotas. The scenario applies to the situation where for a group, institute, or school to be operational, some minimum number of agents must be assigned to that group, institute, or school. An example is Hungary's college admission [4].

In the context of school choice, minimum quotas are often imposed on different **types** of students (e.g., gender, socioeconomic status) [5, 7, 8, 14, 16, 18, 27]. The crucial difference between our setting and these works is that we assume minimum quotas are hard constraints that must be satisfied by any matching, while they treat minimum quotas as "soft" constraints that may or may not actually be satisfied.

Ehlers et al. [8] show that if the constraints are interpreted as hard constraints, no mechanism that is fair and satisfies a definition that they call constrained nonwastefulness can simultaneously be strategy-proof. Due to this impossibility result, Fragiadakis et al. [10] develop two strategy-proof mechanisms that renounce fairness or nonwastefulness. Based on their work, Goto et al. [12] develop a strategy-proof mechanism that can handle hierarchical minimum quotas. We cannot use these mechanisms in our setting since they do not respect initial endowments. Furthermore, they are not Pareto efficient (note that Pareto efficiency implies nonwastefulness but not vice versa).

For school choice programs, fairness is another important criterion, and there exists a trade-off between fairness and efficiency. If the welfare of students is the primary concern, a policymaker should use a Pareto efficient mechanism. For example, a school choice program in New Orleans uses TTC [24]. The uniqueness of our work is that we consider a situation where students are initially endowed with some schools and a trading mechanism is used to achieve Pareto efficiency, while minimum quotas need to be satisfied.

2. MODEL

A market is a tuple $(S, C, X, q_C, p_C, \omega, \succ_S)$.

- $S = \{s_1, \ldots, s_n\}$ is a finite set of students.
- $C = \{c_1, \ldots, c_m\}$ is a finite set of schools.
- $X = S \times C$ is a finite set of contracts. Contract $x = (s, c) \in X$ represents that student s is assigned to school c. For any $X' \subseteq X$, let X'_s denote $\{(s, c) \in X' \mid c \in C\}$, i.e., the sets of contracts related to student s who is involved in X', and let X'_c denote $\{(s, c) \in X' \mid s \in S\}$, i.e., the sets of contracts related to school c involved in X'.
- $q_C = (q_c)_{c \in C}$ is a vector of the schools' maximum quotas.
- $p_C = (p_c)_{c \in C}$ is a vector of the schools' minimum quotas.
- ω: $S \to C$ is an initial endowment function. $\omega(s)$ returns $c \in C$, which is s's initial endowment. When $\omega(s) = c$, we say school c is student s's initial endowment school, and student s is school c's initial endowment student. Let X^* denote $\bigcup_{s \in S}\{(s, \omega(s))\}$, i.e., X^* is the set of contracts, where each element is a contract between a student and her initial endowment school.
- $\succ_S = (\succ_s)_{s \in S}$ is a profile of the students' preferences. For each student s, $\succ_s$ represents the preference of s over X_s. We assume $\succ_s$ is strict for each s. We say (s, c) is **acceptable** for s if $(s, c) \succ_s (s, \omega(s))$ or $c = \omega(s)$ holds. We sometimes use such notations as $c \succ_s c'$ instead of $(s, c) \succ_s (s, c')$.

We assume $\sum_{c \in C} p_c \leq n \leq \sum_{c \in C} q_c$ holds. Also, we assume X^* satisfies minimum/maximum quotas, i.e., for all $c \in C$, $p_c \leq |X^*_c| \leq q_c$ holds.

With a slight abuse of notation, for two sets of contracts, X' and X'', we denote $X'_s \succ_s X''_s$ if either (i) $X'_s = \{x'\}$, $X''_s = \{x''\}$, and $x' \succ_s x''$ for some $x', x'' \in X_s$ that are acceptable for s, or (ii) $X'_s = \{x'\}$ for some $x' \in X_s$ that is acceptable for s and $X''_s = \emptyset$. We denote $X'_s \succeq_s X''_s$ if either $X'_s \succ_s X''_s$ or $X'_s = X''_s$. Also, for $X'_s \subseteq X_s$, we say X'_s is acceptable for s if $X'_s = \{x\}$ and x is acceptable for s.

Definition 1 (feasibility). *$X' \subseteq X$ is **student-feasible** if for all $s \in S$, X'_s is acceptable for s. X' is **school-feasible** if for all $c \in C, p_c \leq |X'_c| \leq q_c$ holds. X' is **feasible** if it is student- and school-feasible.*

We call a feasible set of contracts a **matching**. Note that by definition, any matching is individually rational, i.e., every student is matched with a school that is at least as good as her initial endowment school. Also note that X^* is school-feasible and therefore a matching.

A **mechanism** is a function that takes a profile of students' preferences as input and returns a matching.

Definition 2 (strategy-proofness). *A mechanism is **strategy-proof** if no student ever has any incentive to misreport her preference, regardless of other students' reports.*

Definition 3 (Pareto efficiency). *Matching X' **Pareto dominates** another matching X'' if $\forall s \in S, X'_s \succeq_s X''_s$ and $\exists s \in S, X'_s \succ_s X''_s$ hold, i.e., compared with X'', X' makes all students weakly better off and at least one student strictly better off. A matching is **Pareto efficient** if there is no other matching that Pareto dominates it. A mechanism is Pareto efficient if it always selects a Pareto efficient matching.*

Directed graph G is a pair (V, E) where V is a set of vertices and $E \subseteq \{(i,j) \mid i,j \in V\}$ is a collection of ordered pairs of vertices in V. An ordered pair (i,j), where $i,j \in V$, is called a **directed edge** from i to j.

In a directed graph (V,E), a sequence of vertices $(i_1, \dots, i_k)$, $k \geq 2$, is a **directed path** from vertex i_1 to vertex i_k if $(i_h, i_{h+1}) \in E$ for $h = 1, \dots, k-1$. If $i_1 = i_k$, then we call this directed path a **cycle**. In particular, (i,i), where $(i,i) \in E$, is called a **self-loop** cycle.

3. TOP TRADING CYCLES AMONG REPRESENTATIVES

Before introducing our new mechanism, let us introduce a simpler mechanism based on TTC, which we call Top Trading Cycles among Representatives (TTCR). Since a student is indifferent between multiple seats within the same school, we cannot directly apply the standard TTC mechanism. TTCR is a special case of Algorithm III in [15].

TTCR utilizes $\succ_{ML}$, which is a strict common priority ordering among students called **master list (ML)**. Without loss of generality, we assume $\succ_{ML}$ is defined as follows: $s_1 \succ_{ML} s_2 \succ_{ML} \dots \succ_{ML} s_n$.

This mechanism repeats several rounds. At Round k, Y^{k-1} represents the set of remaining initial endowment contracts and Z represents the set of contracts that have already been finalized. TTCR is defined in Mechanism 1.

Mechanism 1 Top Trading Cycles among Representatives (TTCR)

Initialize $Y^0 = X^*, Z = \emptyset, k = 1$

Round k

Step 1 Create directed graph $G^k = (V^k, E^k)$ as follows:

- V^k is a set of contracts, each of which is selected from each school. More specifically, for each school $c \in C$ s.t. $Y^{k-1}_c \neq \emptyset$, select (s,c) where s has the highest priority among students in Y^{k-1}_c according to the ML.
- E^k is the set of directed edges between contracts in V^k. There exists a directed edge $((s,c),(s',c')) \in E^k$ if c' is the most preferred school according to $\succ_s$ within schools in V^k.

Step 2 Let $\mathscr{C}^k$ denote a set of contracts, each of which is included in a cycle within G^k.

Step 3 For each contract $(s,c) \in \mathscr{C}^k$, let $((s,c),(s',c'))$ denote the direct edge from (s,c). Add (s,c') to Z. $Y^k \leftarrow Y^{k-1} \setminus \mathscr{C}^k$.

Step 4 If $Y^k = \emptyset$, then return Z. Otherwise, $k \leftarrow k+1$ and go to the next round.

Intuitively, we can assume in TTCR, each school chooses one representative student from its initial endowment students based on ML. Then, within these representative students, the standard TTC mechanism is applied. By choosing one representative for each school, we can ignore the fact that a student is indifferent between multiple seats within the same school. Since a student considers her initial endowment school acceptable, there always exists at least one cycle. TTCR can be considered one instance of Algorithm III in [15]. In the algorithm, each school has its own priority ordering among students. Student s, who has the highest priority in school c's ordering, obtains all the seats of c. Then the standard TTC mechanism is applied among the students who own seats. When a student is involved in a cycle and obtains her desired seat, she returns the remaining seats to each school. Then the mechanism repeats the same procedure for the remaining students. If we assume the number of seats available for a school equals the number of its initial endowment students, and school c gives the highest priority to student s according to ML within her initial endowment students, this algorithm becomes identical to the above mechanism.

The obtained matching of TTCR satisfies all minimum/maximum quotas, since for the obtained matching Z, $|Z_c| = |X^*_c|$ holds for all $c \in C$. However, this mechanism is not Pareto efficient, as shown in the following example:

Example 1. *Assume $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}, C = \{c_1, c_2, c_3\}$, where $\omega(s_1) = \omega(s_2) = \omega(s_3) = c_1$, $\omega(s_4) = \omega(s_5) = \omega(s_6) = c_2$, and $\omega(s_7) = c_3$. $q_c = 3$ for all $c \in C$. $p_{c_1} = 2$ and $p_{c_2} = p_{c_3} = 0$.*

The preferences of students are given as follows:

$$\begin{aligned} \succ_{s_1}&: \quad c_2 \quad c_1, \\ \succ_{s_2}, \succ_{s_3}&: \quad c_3 \quad c_1, \\ \succ_{s_4}, \succ_{s_5}, \succ_{s_6}&: \quad c_3 \quad c_2, \\ \succ_{s_7}&: \quad c_1 \quad c_3. \end{aligned}$$

First, Y^0 is determined: $\{(s_1, c_1), (s_2, c_1), (s_3, c_1), (s_4, c_2), (s_5, c_2), (s_6, c_2), (s_7, c_3)\}$.

At Step 1 of Round 1, since $Y_c^0 \neq \emptyset$ for all $c \in C$, the mechanism selects each (s, c) where s has the highest priority according to ML within students in Y_c^0 for all $c \in C$ and adds (s_1, c_1),(s_4, c_2), and (s_7, c_3) to V^1. Then each selected student points to her most preferred school according to $\succ_s$ within schools in V^1; s_1, s_4, and s_7 point to c_2, c_3, and c_1, respectively. Therefore, G^1 is determined as follows:

$$\begin{aligned} V^1 &= \{(s_1, c_1), (s_4, c_2), (s_7, c_3)\}, \\ E^1 &= \{((s_1, c_1), (s_4, c_2)), ((s_4, c_2), (s_7, c_3)), ((s_7, c_3), \\ &\quad (s_1, c_1))\}. \end{aligned}$$

There exists one cycle: $((s_1, c_1), (s_4, c_2), (s_7, c_3), (s_1, c_1))$. At Step 2, $\mathscr{C}^1$ is $\{(s_1, c_1), (s_4, c_2), (s_7, c_3)\}$. At Step 3, (s_1, c_2), (s_4, c_3), and (s_7, c_1) are added to Z and the contracts in $\mathscr{C}^1$ are removed from Y^0. Therefore, Z and Y^1 are determined as follows:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1)\}, \\ Y^1 &= \{(s_2, c_1), (s_3, c_1), (s_5, c_2), (s_6, c_2)\}. \end{aligned}$$

At Step 4, go to Round 2 because $Y^1 \neq \emptyset$.

At Step 1 of Round 2, since $Y_{c_3}^1 = \emptyset$, there is no representative student from c_3. The mechanism selects (s_2, c_1) and (s_5, c_2) according to ML and adds them to V^2. Then each selected student points to her most preferred school according to $\succ_s$ within schools in V^2. Therefore, G^2 is determined as follows:

$$\begin{aligned} V^2 &= \{(s_2, c_1), (s_5, c_2)\}, \\ E^2 &= \{((s_2, c_1), (s_2, c_1)), ((s_5, c_2), (s_5, c_2))\}. \end{aligned}$$

There are two self-loop cycles. At Step 2, $\mathscr{C}^2$ is $\{(s_2, c_1), (s_5, c_2)\}$. Therefore, at Step 3, Z and Y^2 are given as follows:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_1), (s_5, c_2)\}, \\ Y^2 &= \{(s_3, c_1), (s_6, c_2)\}. \end{aligned}$$

At Step 4, go to Round 3 because $Y^2 \neq \emptyset$.

At Step 1 of Round 3, G^3 is determined as follows:

$$\begin{aligned} V^3 &= \{(s_3, c_1), (s_6, c_2)\}, \\ E^3 &= \{((s_3, c_1), (s_3, c_1)), ((s_6, c_2), (s_6, c_2))\}. \end{aligned}$$

There are two self-loop cycles. At Step 2, $\mathscr{C}^2$ is $\{(s_3, c_1), (s_6, c_2)\}$. Therefore, at Step 3, Z and Y^3 are given as follows:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_1), (s_5, c_2), \\ &\quad (s_3, c_1), (s_6, c_2)\}, \\ Y^3 &= \emptyset. \end{aligned}$$

At Step 4, return Z because $Y^3 = \emptyset$.

In the end, obtained matching Z becomes:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_2, c_1), (s_3, c_1), (s_4, c_3), (s_5, c_2), \\ &\quad (s_6, c_2), (s_7, c_1)\}. \end{aligned}$$

Consider another matching Z':

$$\begin{aligned} Z' &= \{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_3), (s_5, c_3), \\ &\quad (s_6, c_2), (s_7, c_1)\}. \end{aligned}$$

We can find that $Z'_s \succeq_s Z_s$ for all $s \in S$ and $Z'_s \succ_s Z_s$ for $s \in \{s_2, s_5\}$ hold. Therefore, Z is not Pareto efficient. Also, $|Z_c| = |X_c^*|$ for all $c \in C$.

4. TOP TRADING CYCLES AMONG REPRESENTATIVES WITH SUPPLEMENTARY SEATS

TTCR's limitation is that it cannot allocate supplementary seats, as shown in the previous example. However, if we allocate supplementary seats too generously, minimum quotas can be violated. In this section, we develop a new Pareto efficient mechanism called Top Trading Cycles among Representatives with Supplementary Seats (TTCR-SS), which utilizes the notion of a dummy student to control supplementary seats at each school. In TTCR-SS, if a school has already "consumed" its initial endowment students and has supplementary seats, it selects a dummy student as its representative.

TTCR-SS repeats several rounds like TTCR. We divide each school c at Round k into the following four categories:

minimum: $|Y_c^{k-1}| > 0$ and $|Z_c| + |Y_c^{k-1}| = p_c$, i.e., c has the remaining initial endowment contracts and the total number of students in the finalized contracts and the initial endowment contracts equals the minimum quota. Thus, a student in its initial endowment contracts cannot move to another school without violating the minimum quota.

decrementable: $|Y_c^{k-1}| > 0$ and $|Z_c| + |Y_c^{k-1}| > p_c$, i.e., c has the remaining initial endowment contracts and a student in its initial endowment contracts can move to another school.

maximum: $|Y_c^{k-1}| = 0$ and $|Z_c| = q_c$, i.e., c has no remaining initial endowment contracts and it has already accepted students up to its maximum quota.

incrementable: $|Y_c^{k-1}| = 0$ and $|Z_c| < q_c$, i.e., c has no remaining initial endowment contract and can accept another student without violating its maximum quota.

Let $C^k_{\min}$, C^k_{dec}, $C^k_{\max}$, and C^k_{inc} represent the sets of schools in each of the above categories, respectively.

TTCR-SS resembles TTCR, but if school c has exhausted its initial endowment students (i.e., $Y_c^{k-1} = \emptyset$ holds), while it has a supplementary seat (i.e., $|Z_c| < q_c$), the school is incrementable and can send dummy student s_d as its representative. If a dummy student points to (s, c) and obtains c's seat, in reality, it means that the number of assigned students in c is decremented by one. To ensure that the obtained matching respects minimum quotas, we carefully design the "preference" of each dummy student. If $|Y_c^{k-1}| + |Z_c| = p_c$ holds for school c, i.e., if c is minimum, then c cannot afford to "accept" a dummy student. Thus, each dummy student points to the contract, in which the student has the highest priority among students whose initial endowment schools are decrementable. Note that all dummy students point to the same contract. Thus, there exists at most one cycle that includes a dummy student. TTCR-SS is defined in Mechanism 2.

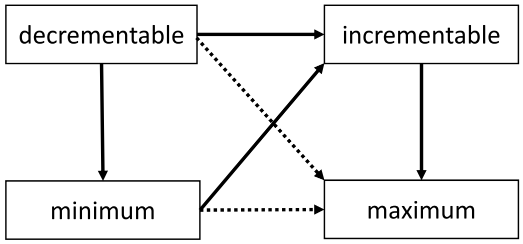

Figure 1: Transition of school categories

Mechanism 2 Top Trading Cycles among Representatives with Supplementary Seats (TTCR-SS)

Initialize $Y^0 = X^*, Z = \emptyset, k = 1$

Round k

Step 1 Create directed graph $G^k = (V^k, E^k)$ as follows:

- V^k is a set of contracts, each of which is selected from each school. More specifically, for each school $c \in C^k_{\min} \cup C^k_{\text{dec}}$, select (s, c) where s has the highest priority among students in Y^{k-1}_c according to ML. Also, for each school $c \in C^k_{\text{inc}}$, select (s_d, c), where s_d is a dummy student, as long as $C^k_{\text{dec}} \neq \emptyset$.
- E^k is the set of directed edges among contracts. There exists a directed edge $((s, c), (s', c')) \in E^k$ if c' is the most preferred school according to $\succ_s$ within schools in V^k. For each contract related to a dummy student (s_d, c), there exists a directed edge $((s_d, c), (s, c')) \in E^k$, where s has the highest priority according to ML within students in V^k and $c' \in C^k_{\text{dec}}$.

Step 2 Let $\mathscr{C}^k$ denote a set of contracts, each of which is included in a cycle within G^k.

Step 3 For each contract $(s, c) \in \mathscr{C}^k$, let $((s, c), (s', c'))$ denote the direct edge from (s, c). Add (s, c') to Z when s is not a dummy student. $Y^k \leftarrow Y^{k-1} \setminus \mathscr{C}^k$.

Step 4 If $Y^k = \emptyset$, then return Z. Otherwise, $k \leftarrow k + 1$ and go to the next round.

Figure 1 shows the possible transition of school categories. Typically, school c is initially decrementable. If $|X^*_c| = p_c$, c is initially minimum. Also, if $|X^*_c| = 0$, c is initially incrementable. As long as all schools are decrementable or minimum, no dummy student is introduced. Thus, for each contract in a cycle, the related school is either decrementable or minimum. Then at some Round k, Y^k_c eventually becomes $\emptyset$ for some school c. c typically becomes incrementable, and a dummy student is introduced. After a dummy student is introduced, for each contract in a cycle, the related school can be incrementable, decrementable, or minimum, and a student whose initial endowment school is decrementable can obtain a seat of an incrementable school from a dummy student. A decrementable school can become minimum, and an incrementable school can become maximum. As an exceptional case (represented by a dotted line in Fig. 1), if for school c, the number of initial endowment students exactly equals q_c, and no student whose initial endowment is c gives a seat to a dummy student, then c becomes maximum when Y^k_c becomes $\emptyset$. As another exceptional case (represented by a dotted line in Fig. 1), if $p_c = q_c$ holds for school c, then c is initially minimum and directly moves to maximum when Y^k_c becomes $\emptyset$. The assignment of maximum schools becomes fixed. When there exits no decrementable school, no dummy student is introduced. Thus, for each contract in a cycle, the related school is minimum. Once this happens, there will be no decrementable school in the later rounds. Thus, no dummy student will be introduced at any later round.

Let us describe how TTCR-SS works.

Example 2. *Consider the same instance as Example 1. Y^0 is the same as Example 1.*

The mechanism behaves exactly the same as the previous example until a dummy student is introduced. The following is the result of Round 1:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1)\}, \\ Y^1 &= \{(s_2, c_1), (s_3, c_1), (s_5, c_2), (s_6, c_2)\}. \end{aligned}$$

At Step 1 of Round 2, c_1 and c_2 are decrementable, and c_3 is incrementable. Schools c_1 and c_2 select their representative students s_2 and s_5, and (s_2, c_1) and (s_5, c_2) are added to V^2. Since there exist decrementable schools, c_3 sends a dummy student and (s_d, c_3) is added to V^2. Then each selected student points to her most preferred school according to $\succ_s$ within schools in V^2; s_2 and s_5 point to c_3. On the other hand, dummy student s_d points to the school whose initial endowment student has the highest priority according to ML within C^2_{dec}; s_d of c_3 points to c_1. Therefore, G^2 is given as follows:

$$\begin{aligned} V^2 &= \{(s_2, c_1), (s_5, c_2), (s_d, c_3)\}, \\ E^2 &= \{((s_2, c_1), (s_d, c_3)), ((s_5, c_2), (s_d, c_3)), ((s_d, c_3), (s_2, c_1))\}. \end{aligned}$$

There is one cycle $((s_2, c_1), (s_d, c_3), (s_2, c_1))$. At Step 2, $\mathscr{C}^2$ is $\{(s_2, c_1), (s_d, c_3)\}$. Therefore, at Step 3, Z and Y^3 are given as follows:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_3)\}, \\ Y^2 &= \{(s_3, c_1), (s_5, c_2), (s_6, c_2)\}. \end{aligned}$$

At Step 1 of Round 3, c_1 is minimum, c_2 is decrementable, and c_3 is incrementable. Thus, the mechanism adds (s_3, c_1), (s_5, c_2), and (s_d, c_3) to V^3. Then, s_3 and s_5 point to c_3. Here, although s_3 has higher priority than s_5 according to the ML, since s_3's initial endowment school c_1 is minimum, s_d points to c_2 instead of c_1. Therefore, G^3 is given as follows:

$$\begin{aligned} V^3 &= \{(s_3, c_1), (s_5, c_2), (s_d, c_3)\}, \\ E^3 &= \{((s_3, c_1), (s_d, c_3)), ((s_5, c_2), (s_d, c_3)), ((s_d, c_3), (s_5, c_2))\}. \end{aligned}$$

There is one cycle $((s_5, c_2), (s_d, c_3), (s_5, c_2))$. At Step 2, $\mathscr{C}^3$ is $\{(s_5, c_2), (s_d, c_3)\}$. Z and Y^3 are given as follows:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_3), (s_5, c_3)\}, \\ Y^3 &= \{(s_3, c_1), (s_6, c_2)\}. \end{aligned}$$

At Step 1 of Round 4, c_1 is minimum, c_2 is decrementable, and c_3 is maximum. Then the mechanism adds (s_3, c_1) and (s_6, c_2) to V^4. Since c_3 is maximum, it cannot send its representative. Thus, no dummy student is added. Therefore, G^4 is given as follows:

$$\begin{aligned} V^4 &= \{(s_3, c_1), (s_6, c_2)\}, \\ E^4 &= \{((s_3, c_1), (s_3, c_1)), ((s_6, c_2), (s_6, c_2))\}. \end{aligned}$$

There are two self-loop cycles. At Step 2, $\mathscr{C}^4$ is $\{(s_3, c_1), (s_6, c_2)\}$. Z and Y^4 are determined as follows:

$$\begin{aligned} Z &= \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_3), (s_5, c_3), \\ &\quad (s_3, c_1), (s_6, c_2)\}, \\ Y^4 &= \emptyset. \end{aligned}$$

At Step 4, since $Y^4 = \emptyset$, the mechanism returns Z.

The obtained matching is identical to Z' in Example 1.

5. THEORETICAL PROPERTIES

In this section, we clarify the theoretical properties of TTCR-SS. We first show its feasibility. Note that there exists at most one cycle that includes a dummy student.

THEOREM 1. *TTCR-SS always produces a feasible matching.*

PROOF. It is clear that the outcome is student-feasible since a student never selects a contract that is related to her unacceptable school.

As for the school-feasibility of Z, we show that $\{Y^k \cup Z\}$ is school-feasible for any k by induction. For $k = 0$, it is clear that $\{Y^0 \cup Z\}$ is school-feasible because $Y^0 = X^*$ is school-feasible and $Z = \emptyset$. Suppose $\{Y^k \cup Z\}$ is school-feasible for some k. The induction completes if we show that for any $c \in C$ it holds that $p_c \leq |Y_c^{k+1}| + |Z_c| \leq q_c$ after Round $k+1$. If a contract related to c is not included in $\mathscr{C}^{k+1}$, the assignment related to c never changes. Thus, assume (s, c) is included in $\mathscr{C}^{k+1}$. Then, it is clear that c is not maximum. If c is incrementable, $Y_c^k = Y_c^{k+1} = \emptyset$, and $p_c \leq |Z_c| < q_c$ holds at the beginning of Round $k+1$. Z_c is incremented by one. Thus, $p_c \leq |Z_c| \leq q_c$ holds at the end of Round $k+1$. If c is decrementable, $p_c < |Y_c^k| + |Z_c| \leq q_c$ holds at the beginning of Round $k+1$. Also, $|Y_c^k| - 1 = |Y_c^{k+1}|$ holds, and Z_c does not change if a dummy student obtains the seat of c or is incremented by one if a non-dummy student obtains the seat of c. In either case, $p_c \leq |Y_c^{k+1}| + |Z_c| \leq q_c$ holds at the end of Round $k+1$. If c is minimum, $p_c = |Y_c^k| + |Z_c| \leq q_c$ holds at the beginning of Round $k+1$. $|Y_c^k| - 1 = |Y_c^{k+1}|$ holds, and Z_c is always incremented by one since a dummy student never obtains a seat of c. Thus, $p_c = |Y_c^{k+1}| + |Z_c| \leq q_c$ holds at the end of Round $k+1$. □

School c is **available** at Round k if either $C_{\text{dec}}^k \neq \emptyset$ and $c \in C \setminus C_{\text{max}}^k$ or $C_{\text{dec}}^k = \emptyset$ and $c \in C_{\text{min}}^k$ hold. Let C_{ava}^k denote the set of all available schools. It is clear that at Round k, a contract related to school c is included in V^k if and only if c is available at Round k.

It is obvious that the following lemma holds from the category transition of schools and the definition of C_{ava}^k.

LEMMA 1. *For any two rounds k and k' with $k < k'$, $C_{dec}^k \supseteq C_{dec}^{k'}$ and $C_{ava}^k \supseteq C_{ava}^{k'}$ hold.*

Intuitively, this lemma means that the possible choices for a student weakly monotonically shrinks in the later rounds. As a result, the following lemma says the choice of a student is the best within all schools that are available in the later rounds.

LEMMA 2. *Suppose TTCR-SS obtains X'. For any k and any $c \in C_{ava}^k$, and any student s who is included in $\mathscr{C}^{k'}$, i.e., a cycle at Round $k' \leq k$, $X_s' \succeq_s \{(s, c)\}$ holds.*

PROOF. From Lemma 1, $C_{\text{ava}}^k \subseteq C_{\text{ava}}^{k'}$ holds. Also, the fact that s is included in $\mathscr{C}^{k'}$ means that $\{(s, c')\} = X_s'$ and c' is the most preferred school for s within $C_{\text{ava}}^{k'}$. Thus, $X_s' \succeq_s \{(s, c)\}$ holds. □

The following lemma implies if there exists a directed path toward a contract at some round, the path remains in the later rounds unless the contract is removed by being included in a cycle. Thus, if a student can obtain a seat of a particular school in a round (either by truth-telling or by manipulation), she can also obtain the seat in the later rounds.

LEMMA 3. *Suppose there is a directed path from a contract (s, c) to (s', c') in G^k, and suppose $(s', c') \in V^{k'}$ for some $k' > k$. Then exactly the same directed path from (s, c) to (s', c') exists in $E^{k'}$.*

PROOF. It is sufficient to show that $((s, c), (s', c')) \in E^k$ and $(s', c') \in V^{k+1}$ imply $((s, c), (s', c')) \in E^{k+1}$, since a directed path is a sequence of directed edges. First, suppose $s \neq s_d$. From Lemma 1, $C_{\text{ava}}^{k+1} \subseteq C_{\text{ava}}^k$ holds. Since c' is the most preferred school for s within C_{ava}^k, if $c' \in C_{\text{ava}}^{k+1}$, c' remains the most preferred school for s within C_{ava}^{k+1}. Thus, $((s, c), (s', c')) \in E^{k+1}$ holds. Second, suppose $s = s_d$. The fact that $((s_d, c), (s', c')) \in E^k$ implies that c' is decrementable and that s' has the highest priority in ML within all the remaining students in all the decrementable schools. According to Lemma 1, the set of decrementable schools never expands. As long as (s', c') remains in V^{k+1}, c' remains decrementable and s' still has the highest priority in ML within all the remaining students in all the decrementable schools at Round $k+1$. Thus, $((s, c), (s', c')) \in E^{k+1}$ holds. □

The following lemma means that the declared preference of a student does not affect the outcome of the rounds before she is included in a cycle.

LEMMA 4. *Fix the reported preferences of all students except s at $\succ_{-s} = (\succ_{s'})_{s' \in S \setminus \{s\}}$. Suppose that $(s, \omega(s)) \in \mathscr{C}^k$ if she reports $\succ_s$ and $(s, \omega(s)) \in \mathscr{C}^{k'}$ if she reports $\succ_s'$, where $k \leq k'$. Then C_*^k, where "$*$" can be either "max", "min", "inc", "dec", or "ava", does not change regardless of whether student s reports $\succ_s$ or $\succ_s'$.*

PROOF. Since $(s, \omega(s)) \notin \mathscr{C}^{\hat{k}}$ holds for any $\hat{k} < k$, the same contracts form cycles before Round k whether student s reports $\succ_s$ or $\succ_s'$. Then in both cases, the same contracts remain in Y^{k-1} and the same contracts are added to Z, which implies C_*^k does not change. □

Now, we are ready to prove that TTCR-SS is strategy-proof and Pareto efficient.

THEOREM 2. *TTCR-SS is strategy-proof.*

PROOF. Fix the reported preferences of all students except s at $\succ_{-s} = (\succ_{s'})_{s' \in S \setminus \{s\}}$ and denote $\succ = (\succ_{-s}, \succ_s)$ and $\succ' = (\succ_{-s}, \succ_s')$, where $\succ_s$ is her true preference and $\succ_s'$ is a fake preference. For some Round k, explicitly write $V^k(\succ)$, $G^k(\succ)$, $E^k(\succ)$, and $\mathscr{C}^k(\succ)$ to denote V^k, G^k, E^k, and $\mathscr{C}^k$ when the reported preference profile is $\succ$ and so on. Explicitly write $C_*^k(\succ)$ to denote C_*^k when the reported preference profile is $\succ$ and so on. Suppose that $(s, \omega(s)) \in \mathscr{C}^k(\succ)$, i.e., if s reports her true preference $\succ_s$, she belongs to a cycle at

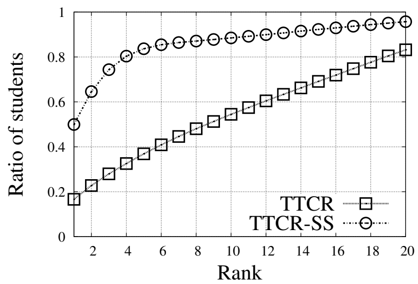

Figure 2: CDFs of students' welfare ($\alpha = 0.6$)

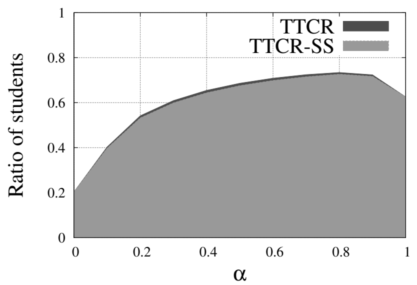

Figure 3: Ratio of students who prefer TTCR-SS/TTCR

Round k, and $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$, i.e., if she reports some other preference $\succ'_s$, she belongs to a cycle at Round k'.

First, assume $k \leq k'$. Since $(s, \omega(s)) \in \mathscr{C}^k(\succ)$, s must be matched with her most preferred school within $C^k_{\text{ava}}(\succ)$. Also, $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$ means that s is matched with a school within $C^{k'}_{\text{ava}}(\succ')$. Therefore, it is sufficient to show that $C^{k'}_{\text{ava}}(\succ') \subseteq C^k_{\text{ava}}(\succ)$ holds. Since $k \leq k'$, it follows from Lemma 4 that $C^k_{\text{ava}}(\succ') = C^k_{\text{ava}}(\succ)$ holds. Also, from Lemma 1, it follows that $C^{k'}_{\text{ava}}(\succ') \subseteq C^k_{\text{ava}}(\succ')$. Combining these results, we have $C^{k'}_{\text{ava}}(\succ') \subseteq C^k_{\text{ava}}(\succ)$.

Next, assume $k > k'$. Since $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$, there exists a directed path from (s', c') to $(s, \omega(s))$ in $G^{k'}(\succ')$, where $((s, \omega(s)), (s', c')) \in E^{k'}(\succ')$. From Lemma 4, $C^{k'}_{\text{ava}}(\succ') = C^{k'}_{\text{ava}}(\succ)$ holds. Thus, there exists the same directed path from (s', c') to $(s, \omega(s))$ in $G^{k'}(\succ)$. The fact that $(s, \omega(s)) \in \mathscr{C}^k(\succ)$ implies $(s, \omega(s)) \in V^k(\succ)$, and thus from Lemma 3 there exists the same directed path from (s', c') to $(s, \omega(s))$ in $G^k(\succ)$. Then s's assignment under $\succ$ is at least as good as c', which is the assignment under $\succ'$. Thus, s cannot be better off by reporting $\succ'_s$. $\square$

THEOREM 3. *TTCR-SS is Pareto efficient.*

PROOF. Let X' denote the matching produced by TTCR-SS and suppose for contradiction that X' can be Pareto improved, i.e., there exists another matching X'' that Pareto dominates X'. Let S' denote the set of students such that $s \in S'$ if and only if $X''_s \succ_s X'_s$ holds, i.e., students whose assignments are strictly improved in X''.

Among the students in S', let $\widehat{s}_1$ be a student who is lastly allocated under the mechanism when generating X', i.e., $(\widehat{s}_1, \omega(\widehat{s}_1)) \in \mathscr{C}^{k_1}$ for some k_1 and there exists no $s' \in S'$ such that $(s', \omega(s')) \in \mathscr{C}^{k'}$ with $k' > k_1$. Since $\widehat{s}_1 \in S'$, it follows that $k_1 > 1$. If $k_1 = 1$ holds, $X'_{\widehat{s}_1}$ is the contract between $\widehat{s}_1$ and her most preferred school in C, and therefore no contract can improve $\widehat{s}_1$ from $X'_{\widehat{s}_1}$, which contradicts that $\widehat{s}_1 \in S'$.

Let $\widehat{c}_1$ be the school to which $\widehat{s}_1$ is assigned under X', i.e., $\{(\widehat{s}_1, \widehat{c}_1)\} = X'_{\widehat{s}_1}$. Since $\widehat{c}_1 \in C^{k_1}_{\text{ava}}$, Lemma 2 implies that for all $s \in S'$, it holds that $X'_s \succeq_s \{(s, \widehat{c}_1)\}$. Therefore, $X''_s \neq \{(s, \widehat{c}_1)\}$ for all $s \in S'$, and together with the fact that $X'_{\widehat{s}_1} = \{(\widehat{s}_1, \widehat{c}_1)\} \notin X''$, it holds that $X''_{\widehat{c}_1} \subsetneq X'_{\widehat{c}_1}$. Since X'' is feasible and thus $p_{\widehat{c}_1} \leq |X''_{\widehat{c}_1}|$, it holds that $p_{\widehat{c}_1} < |X'_{\widehat{c}_1}|$.

Let $\widehat{c}_2$ be the school to which $\widehat{s}_1$ is assigned under X'', i.e., $\{(\widehat{s}_1, \widehat{c}_2)\} = X''_{\widehat{s}_1}$. By the definition of X'', it must hold that $\widehat{c}_2 \succ_{\widehat{s}_1} \widehat{c}_1$. Since $\{(\widehat{s}_1, \widehat{c}_1)\} = X'_{\widehat{s}_1}$ and $(\widehat{s}_1, \omega(\widehat{s}_1)) \in \mathscr{C}^{k_1}$, there exists a directed edge $((\widehat{s}_1, \omega(\widehat{s}_1)), (s, \widehat{c}_1)) \in E^{k_1}$. The fact that $\widehat{s}_1$ points to $\widehat{c}_1$ even though she prefers $\widehat{c}_2$ over $\widehat{c}_1$ implies either (i) $\widehat{c}_2$ is maximum at Round k_1 holds or (ii) $\widehat{c}_2$ is incrementable and there exists no decrementable school at Round k_1 holds. However, in this case, (ii) cannot hold; if it holds, all the contracts in V^{k_1} are related to minimum schools, but $\widehat{c}_1$ is not minimum. Thus, $\widehat{c}_2$ must be maximum at Round k_1. Then $|X'_{\widehat{c}_2}| = q_{\widehat{c}_2}$ holds. Since X'' is feasible and $(\widehat{s}_1, \widehat{c}_2) \notin X'$ but $(\widehat{s}_1, \widehat{c}_2) \in X''$, there must exist student $\widehat{s}_2$ who is matched with $\widehat{c}_2$ in X' and her assignment is changed in X'', i.e., $\widehat{s}_2 \in S'$ and $(\widehat{s}_2, \widehat{c}_2) \in X'$, and suppose $(\widehat{s}_2, \omega(\widehat{s}_2)) \in \mathscr{C}^{k_2}$ for some k_2, i.e., $\widehat{s}_2$ belongs to a cycle and is matched to $\widehat{c}_2$ at Round k_2. Then $\widehat{c}_2 \in C^{k_2}_{\text{ava}}$ must hold and Lemma 1 implies that $k_1 > k_2$.

Let $\widehat{c}_3$ be the school to which $\widehat{s}_2$ is assigned under X'', i.e., $\{(\widehat{s}_2, \widehat{c}_3)\} = X''_{\widehat{s}_2}$. Then by the definition of X'', it holds that $\widehat{c}_3 \succ_{\widehat{s}_2} \widehat{c}_2$. Since $\{(\widehat{s}_2, \widehat{c}_2)\} = X'_{\widehat{s}_2}$, A directed edge exists $((\widehat{s}_2, \omega(\widehat{s}_2)), (s, \widehat{c}_2)) \in E^{k_2}$. The fact that $\widehat{s}_2$ points to $\widehat{c}_2$ even though she prefers $\widehat{c}_3$ over $\widehat{c}_2$ implies that either $\widehat{c}_3$ is maximum or no decrementable schools exist. Using the similar argument as $\widehat{s}_1$, we can derive that $\widehat{c}_3$ is maximum and $|X'_{\widehat{c}_3}| = q_{\widehat{c}_3}$ holds. Since X'' is feasible and $(\widehat{s}_2, \widehat{c}_3) \notin X'$ but $(\widehat{s}_2, \widehat{c}_3) \in X''$, there must exist a student who is matched with $\widehat{c}_3$ in X' and her assignment is changed in X''. Let $\widehat{s}_3$ be such a student, i.e., $(\widehat{s}_3, \widehat{c}_3) \in X'$ and $\widehat{s}_3 \in S'$, and suppose $(\widehat{s}_3, \omega(\widehat{s}_3)) \in \mathscr{C}^{k_3}$ for some k_3, i.e., $\widehat{s}_3$ belongs to a cycle and is matched to $\widehat{c}_3$ at Round k_3. Then, by the same argument showing $k_1 > k_2$, it holds that $k_2 > k_3$.

We can continue this argument until we find a student $\widehat{s}_* \in S'$ with $(\widehat{s}_*, \omega(\widehat{s}_*)) \in \mathscr{C}^1$, i.e., a student who belongs to a cycle at Round 1 and whose assignment is strictly improved under X''. However, $\widehat{s}_*$ is matched with her most preferred school in C under X' and there is no contract for $\widehat{s}_*$ that is strictly better than $X'_{\widehat{s}_*}$, which contradicts that $\widehat{s}_* \in S'$ and thus no such X'' exists. $\square$

The fact that TTCR-SS is Pareto efficient, while TTCR is not, does not imply that all students weakly prefer the matching of TTCR-SS over that of TTCR. This is because there can be multiple Pareto efficient matchings. Let us show a simple example. Assume there are two students, s_1 and s_2, and three schools, c_1, c_2, and c_3. The minimum quota of c_1 is 1, and the minimum quotas of c_2 and c_3 are 0. The maximum quotas of all the schools are 1. The initial endowment schools of s_1 and s_2 are c_1 and c_2, respectively. The preference of s_1 is $c_2 \succ_{s_1} c_1$, and the preference of s_2 is $c_3 \succ_{s_2} c_1 \succ_{s_2} c_2$. In TTCR-SS, the dummy student of c_3 and s_2 swap their seats, and s_1 cannot move to c_2. Thus, the obtained matching is $\{(s_1, c_1), (s_2, c_3)\}$. On the other hand, in TTCR, no dummy student is introduced and s_1 and s_2 swap their seats. The obtained matching is

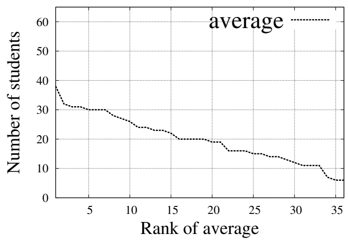

Figure 4: Difference of number of assigned students ($\alpha = 0$)

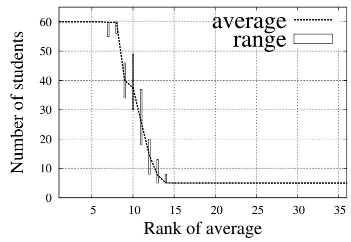

Figure 5: Difference of number of assigned students ($\alpha = 0.6$)

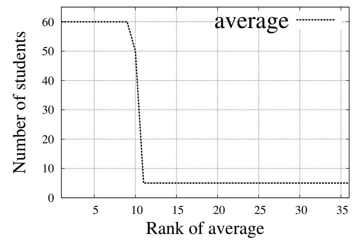

Figure 6: Difference of number of assigned students ($\alpha = 1$)

$\{(s_1, c_2), (s_2, c_1)\}$. Here, s_1 prefers the matching of TTCR. In the next section, we experimentally show that the overwhelming majority of students prefer the matching obtained by TTCR-SS.

Finally, we show that TTCR-SS can be done in polynomial time in $|S|$ and $|C|$.

THEOREM 4. *The time complexity of TTCR-SS is* $O(|S| \cdot |C|)$.

PROOF. At each round, there exists at least one cycle. A cycle must contain at least one student $s \in S$, and the assignment of s is fixed. Thus, the number of rounds required for TTCR-SS is at most $|S|$. Also, for each round, there are at most $|C|$ contracts, and finding cycles can be done in $O(|C|)$. Therefore, the time complexity of TTCR-SS is $O(|S| \cdot |C|)$. □

6. EVALUATION

This section evaluates TTCR-SS. We consider a market with $n = 720$ students and $m = 36$ schools. Each school has 20 initial endowment students. The minimum and maximum quotas of each school are 5 and 60, respectively. We generate students' preferences as follows. We draw one common vector v of the cardinal utilities from set $[0, 1]^m$ uniformly at random. We then randomly draw private vector u_s of the cardinal utilities from the same set, again uniformly at random. Next, we construct cardinal utilities over all m schools for student s as $\alpha v + (1 - \alpha)u_s$ for some $\alpha \in [0, 1]$. We convert these cardinal utilities into an ordinal preference relation for each student. The higher the value of α is, the more correlated the student preferences are.

6.1 Comparison with TTCR

We compare the welfare of the students for TTCR and TTCR-SS. We generate 100 problem instances. ML is set to $s_1, \ldots, s_n$. Figure 2 shows the Cumulative Distribution Functions (CDFs) of the average number of students matched with their k-th or higher ranked school under each mechanism when α is 0.6. Hence, a steep upper trend line is desirable. 50% of the students obtain their first choices, and 65% obtain their first or second choices in TTCR-SS, while in TTCR, only 16% of students obtain their most preferred contracts, and 23% obtain their most or second preferred contracts. Thus, the average student welfare of TTCR-SS clearly outperforms that of TTCR.

Next, we show the average ratio of students who strictly prefer the matching of one mechanism over that of the other mechanism (Fig. 3). Here, the gray area shows the ratio of students who prefer the matching of TTCR-SS, while the black area shows the ratio of students who prefer the matching of TTCR (note that the area is very narrow and it looks like a line). The white area shows the ratio of students whose assignments are the same. For example, when $\alpha = 0.6$, only 1% of the students prefer the matching of TTCR, while 70% prefer the matching of TTCR-SS; the overwhelming majority of students prefer the matching obtained by TTCR-SS.

6.2 Effect of choice of master list

ML must be chosen exogenously by the mechanism. If the obtained matching can vary significantly according to the choice of the ML, say, the number of students assigned to each school changes significantly, how to determine ML can be controversial. Here, we fix one problem instance and compare the results of TTCR-SS for 100 randomly generated different MLs. Figures 4, 5, and 6 show differences of the number of assigned students under the cases of $\alpha = 0$, 0.6, and 1, respectively. We show the average, minimum, and maximum of the number of allocated students for each school. The x-axis represents schools that are sorted in decreasing order of their average. When $\alpha = 0$, the preference of students are independent and there is virtually no competition among them. Then the choice of ML does not affect the outcome very much; the average, minimum, and maximum are almost the same (Fig. 4). When $\alpha = 1$, the preferences of the students are the same and they all compete for the seats of the same popular schools. Thus, the choice of ML affects who will be assigned to the popular schools, but it does not affect the number of students assigned to them. Thus, the average, minimum, and maximum are identical (Fig. 6). When $\alpha = 0.6$, the number of allocated students can vary according to the choice of ML. However, Figure 5 shows that the numbers of students allocated to popular/unpopular schools are almost the same, and it varies in the schools that are in the middle of popular/unpopular schools. Thus, we conjecture that the choice of ML is not too controversial; the choice does not affect the popularity of schools very much.

7. CONCLUSIONS

In this paper, we developed TTCR-SS for a school choice program, where the obtained matching must respect minimum quotas and initial endowments. We proved that TTCR-SS is strategy-proof and Pareto efficient. Our simulation results showed that it is significantly better than TTCR, which cannot allocate supplementary seats.

Our immediate future work is to extend our mechanism to handle different types of distributional constraints besides minimum quotas [17].

REFERENCES

[1] A. Abdulkadiroğlu and Y.-K. Che. The role of priorities in assigning indivisible objects: A characterization of top trading cycles. Working Paper, 2010.

[2] A. Abdulkadiroğlu and T. Sönmez. House allocation with existing tenants. *Journal of Economic Theory*, 88:233–260, 1999.

[3] A. Abdulkadiroğlu and T. Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

[4] P. Biró, T. Fleiner, R. W. Irving, and D. F. Manlove. The college admission problem with lower and common quotas. *Theoretical Computer Science*, 411:3136 – 3153, 2010.

[5] S. Braun, N. Dwenger, D. Kübler, and A. Westkamp. Implementing quotas in university admissions: An experimental analysis. *Games and Economic Behavior*, 85:232–251, 2014.

[6] U. M. Dur. A characterization of the top trading cycles mechanism for school choice problem. Working Paper, 2013.

[7] U. M. Dur and M. U. Ünver. Two-sided matching via balanced exchange: Tuition and worker exchanges. Working Paper, 2015.

[8] L. Ehlers, I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim. School choice with controlled choice constraints: Hard bounds versus soft bounds. *Journal of Economic Theory*, 153:648–683, 2014.

[9] A. Erdil and H. Ergin. What's the matter with tie-breaking? improving efficiency in school choice. *The American Economic Review*, 98(3):669–689, 2008.

[10] D. Fragiadakis, A. Iwasaki, P. Troyan, S. Ueda, and M. Yokoo. Strategyproof matching with minimum quotas. *ACM Transactions on Economics and Computation*, 4(1):6:1–6:40, 2015.

[11] D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[12] M. Goto, N. Hashimoto, A. Iwasaki, Y. Kawasaki, S. Ueda, Y. Yasuda, and M. Yokoo. Strategy-proof matching with regional minimum quotas. In *Proceedings of Thirteenth International Conference on Autonomous Agents and Multiagent Systems (AAMAS-2014)*, pages 1225–1232, 2014.

[13] G. Haeringer and F. Klijn. Constrained school choice. *Journal of Economic Theory*, 144:1921 – 1947, 2009.

[14] I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim. Effective affirmative action in school choice. *Theoretical Economics*, 8(2):325–363, 2013.

[15] O. Kesten. On two competing mechanisms for priority-based allocation problems. *Journal of Economic Theory*, 127:155 – 171, 2006.

[16] F. Kojima. School choice: Impossibilities for affirmative action. *Games and Economic Behavior*, 75(2):685–693, 2012.

[17] F. Kojima, A. Tamura, and M. Yokoo. Designing matching mechanisms under constraints: An approach from discrete convex analysis. In *Proceedings of the Seventh International Symposium on Algorithmic Game Theory (SAGT-2014)*, 2014. the full version is available at http://mpra.ub.uni-muenchen.de/62226).

[18] R. Kurata, M. Goto, A. Iwasaki, and M. Yokoo. Controlled school choice with soft bounds and overlapping types. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence (AAAI-2015)*, pages 951–957, 2015.

[19] J. Ma. Strategy-proofness and the strict core in a market with indivisibilities. *International Journal of Game Theory*, 23:75 – 83, 1994.

[20] T. Morrill. An alternative characterization of top trading cycles. *Economic Theory*, 54:181 – 197, 2013.

[21] S. Pápai. Strategyproof assignment by hierarchical exchange. *Econometrica*, 68(6):1403–1433, 2000.

[22] M. Pycia and M. U. Ünver. Incentive compatible allocation and exchange of discrete resources. *Theoretical Economics*, 2016. Forthcoming.

[23] A. E. Roth. Incentive compatibility in a market with indivisible goods. *Economics Letters*, 9:127 – 132, 1982.

[24] A. E. Roth. School choice in New Orleans with top trading cycles, April 17 2012. http://marketdesigner.blogspot.jp/2012/04/school-choice-in-new-orleans-with-top.html.

[25] A. E. Roth and A. Postlewaite. Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics*, 4:131 – 137, 1977.

[26] L. Shapley and H. Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1:23 – 37, 1974.

[27] A. Westkamp. An analysis of the German university admissions system. *Economic Theory*, 53:561–589, 2013.