Approval Voting with Intransitive Preferences (Extended Abstract)

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ABSTRACT

We extend Approval voting to the settings where voters may have intransitive preferences. The major obstacle to applying Approval voting in these settings is that voters are not able to clearly determine who they should approve or disapprove, due to the intransitivity of their preferences. An approach to address this issue is to apply tournament solutions to help voters make the decision. We study a class of voting systems where first each voter casts a vote defined as a tournament, then a well-defined tournament solution is applied to select the candidates who are assumed to be approved by the voter. Winners are the ones receiving the most approvals. We study axiomatic properties of this class of voting systems and complexity of control and bribery problems for these voting systems.

Keywords

approval; tournament solution; voting system; complexity

1. INTRODUCTION

Approval-based voting systems are among the most important voting systems and have been extensively studied in the literature [1, 2, 7, 8, 9, 13]. There are real-world applications where voters have intransitive preferences [4, 5, 6, 10, 11]. We extend the framework of approval-based voting to the settings where voters may hold intransitive preferences over the candidates, by combining approval voting and tournament solutions. In particular, each voter submits a vote defined as a tournament, and approves exactly the candidates selected by a certain tournament solution. The winners are the candidates with the most approvals. We first study some axiomatic properties of these voting systems (Theorems 1-4). In addition, we study the complexity of control and bribery problems for these voting systems. Table 1 summarizes these complexity results. See [12] for a full version of this paper.

A *tournament* T is a pair $(V(T), \succ)$ where $V(T)$ is a set of *candidates* and $\succ$ is an asymmetric and complete binary relation on $V(T)$. For $a \in V(T)$, $N^-_T(a) = \{b \in V(T) \mid b \succ a\}$ and $N^+_T(a) = \{b \in V(T) \mid a \succ b\}$. A candidate a is the *source* (Condorcet winner) of T if $N^-_T(a) = \emptyset$. For $B \subseteq V(T)$, $T[B]$ is the *subtournament* induced by B, i.e., $T[B] = (B, \succ')$ where for every $a, b \in B$, $a \succ' b$ if and only if $a \succ b$. A *tournament solution* π is a function that maps every tournament T to a nonempty subset $\pi(T) \subseteq V(T)$. In this paper, we mainly study the top cycle (TC), Copeland set (CO) and uncovered set (UC). See [3] for the definitions of these tournament solutions.

An *election* is a tuple $\mathcal{E} = (\mathcal{C}, \mathcal{T})$, where $\mathcal{C}$ is a set of candidates, and $\mathcal{T}$ is a list of *votes* defined as tournaments. A *voting correspondence* φ is a function that maps an election $\mathcal{E} = (\mathcal{C}, \mathcal{T})$ to a nonempty subset $\varphi(\mathcal{E})$ of $\mathcal{C}$. We call the elements in $\varphi(\mathcal{E})$ the *winners* of $\mathcal{E}$ with respect to φ. For two non-overlapping lists of tournaments $\mathcal{T} = (T_1, T_2, ..., T_x)$ and $\mathcal{T}' = (T'_1, T'_2, ..., T'_y)$, $\mathcal{T}+\mathcal{T}'$ is the list $(T_1, ..., T_x, T'_1, ..., T'_y)$. For two elections $\mathcal{E} = (\mathcal{C}, \mathcal{T})$ and $\mathcal{E}' = (\mathcal{C}, \mathcal{T}')$ with the same candidate set $\mathcal{C}$, $\mathcal{E}+\mathcal{E}' = (\mathcal{C}, \mathcal{T}+\mathcal{T}')$.

Now we introduce the core concept in this paper—π-Approval.

π-Approval

Each candidate $c \in \mathcal{C}$ is assigned a score defined as $score(c, \mathcal{E}, \pi) = |\{T \in \mathcal{T} \mid c \in \pi(T)\}|$. The candidates with the highest score are the winners.

Properties of Voting Correspondences φ.

Anonymity: for every two elections $\mathcal{E} = (\mathcal{C}, \mathcal{T} = (T_1, ..., T_n))$ and $\mathcal{E}' = (\mathcal{C}, \mathcal{T}' = (T_{\sigma(1)}, ..., T_{\sigma(n)}))$ where $(\sigma(1), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$, it holds that $\varphi(\mathcal{E}) = \varphi(\mathcal{E}')$.

Neutrality: An election $(\mathcal{C}, \mathcal{T} = (T_1, ..., T_n))$ is *isomorphic* to another election $(\mathcal{C}', \mathcal{T}' = (T'_1, ..., T'_n))$ where $T_i = (\mathcal{C}, \succ_i)$ and $T'_i = (\mathcal{C}', \succ'_i)$ for every $i \in \{1, ..., n\}$, if there is an one-to-one mapping $f : \mathcal{C} \mapsto \mathcal{C}'$ such that for every two distinct candidates $a, b \in \mathcal{C}$ and every $i \in \{1, 2, ..., n\}$, it holds that $a \succ_i b$ if and only if $f(a) \succ'_i f(b)$. A voting correspondence φ is neutral if for every two isomorphic elections $\mathcal{E} = (\mathcal{C}, \mathcal{T})$ and $\mathcal{E}' = (\mathcal{C}', \mathcal{T}')$, and every $c \in \mathcal{C}$, it holds that $c \in \varphi(\mathcal{E})$ if and only if $f(c) \in \varphi(\mathcal{E}')$, where f is the mapping as discussed above for $\mathcal{E}$ and $\mathcal{E}'$.

Monotonicity: for every two elections $\mathcal{E} = (\mathcal{C}, \mathcal{T} = (T_1, ..., T_n))$ and $\mathcal{E}' = (\mathcal{C}, \mathcal{T}' = (T'_1, ..., T'_n))$, and every $c \in \varphi(\mathcal{E})$ such that for every $i \in \{1, 2, ..., n\}$ (1) $T_i[\mathcal{C} \setminus \{c\}] = T'_i[\mathcal{C} \setminus \{c\}]$; and (2) $N^+_{T_i}(c) \subseteq N^+_{T'_i}(c)$, it holds that $c \in \varphi(\mathcal{E}')$.

Majority: for every election $\mathcal{E} = (\mathcal{C}, \mathcal{T})$ where there is a candidate $c \in \mathcal{C}$ which is the source in a majority of the tournaments in $\mathcal{T}$, it holds that $c \in \varphi(\mathcal{E})$.

Consistency: for every two elections $\mathcal{E} = (\mathcal{C}, \mathcal{T})$ and $\mathcal{E}' = (\mathcal{C}, \mathcal{T}')$, it holds that $\varphi(\mathcal{E}) \cap \varphi(\mathcal{E}') \subseteq \varphi(\mathcal{E} + \mathcal{E}')$.

Pareto optimal: for every election $\mathcal{E} = (\mathcal{C}, \mathcal{T})$ and every two candidates $a, b \in \mathcal{C}$ such that $a \succ b$ in every tournament $T = (\mathcal{C}, \succ) \in \mathcal{T}$, $a \notin \varphi(\mathcal{E})$ implies $b \notin \varphi(\mathcal{E})$.

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Properties of Tournament Solutions π. The prefix "TS" in the following notations is to avoid confusion with the axiomatic properties of voting correspondences.

TS-Neutrality: Two tournaments $T = (\mathcal{C}, \succ)$ and $T' = (\mathcal{C}', \succ')$ where $|\mathcal{C}| = |\mathcal{C}'|$ are *isomorphic* if there is an one-to-one mapping $f : \mathcal{C} \mapsto \mathcal{C}'$ such that for every $a, b \in \mathcal{C}$, it holds that $a \succ b$ if and only if $f(a) \succ' f(b)$. Here, f is called an *isomorphic mapping* of T and T'. A tournament solution π satisfies the TS-neutrality criterion if for every two isomorphic tournaments $T = (\mathcal{C}, \succ)$ and $T' = (\mathcal{C}', \succ')$, it holds that $\pi(T') = \{f(a) \in \mathcal{C}' \mid a \in \pi(T)\}$, where f is an isomorphic mapping of T and T'.

TS-Monotonicity: for every $T = (\mathcal{C}, \succ)$, $T' = (\mathcal{C}, \succ')$, and $c \in \pi(T)$ such that $T[\mathcal{C} \setminus \{c\}] = T'[\mathcal{C} \setminus \{c\}]$ and $N^+_T(c) \subseteq N^+_{T'}(c)$, it holds that $c \in \pi(T')$.

TS-Condorcet consistency: for every tournament T which has a source w, it holds that $\pi(T) = \{w\}$.

To the best of our knowledge, the following two concepts of monotonocity have not been studied in the literature.

TS-Exclusive monotonicity: for every $T = (\mathcal{C}, \succ)$, $T' = (\mathcal{C}, \succ')$, and $c \in \pi(T)$ with $T[\mathcal{C} \setminus \{c\}] = T'[\mathcal{C} \setminus \{c\}]$ and $N^+_T(c) \subseteq N^+_{T'}(c)$, it holds that $c \in \pi(T')$ and $\pi(T') \subseteq \pi(T)$.

TS-Exclusive negative monotonicity (TS-ENM): for every two tournaments $T = (\mathcal{C}, \succ)$ and $T' = (\mathcal{C}, \succ')$, and every $c \notin \pi(T)$ such that (1) $T[\mathcal{C} \setminus \{c\}] = T'[\mathcal{C} \setminus \{c\}]$; and (2) $N^+_T(c) \subseteq N^+_{T'}(c)$, it holds that $\pi(T') \nsubseteq \pi(T)$ implies $c \in \pi(T')$.

2. AXIOMATIC PROPERTIES

It is fairly easy to check that π-Approval is anonymous for all tournament solutions π. Moreover, π-Approval is neutral for all tournament solutions π which satisfy the TS-neutrality criterion. Furthermore, π-Approval satisfies the majority criteria for all π that are TS-Condorcet consistent. We now study some other properties for π-Approval. Consider first the consistency criterion.

THEOREM 1. *π-Approval is consistent for all tournament solutions π.*

Now we study the monotonicity of π-Approval for all TS-Condorcet consistent tournament solutions π. Many commonly used tournament solutions including all tournament solutions studied in this paper are TS-Condorcet consistent. We derive both sufficient and necessary conditions for such π-Approval to be monotonic.

THEOREM 2. *Let π be a TS-Condorcet consistent tournament solution. Then, π-Approval is monotonic if and only if π satisfies the TS-exclusive monotonicity and TS-ENM criteria.*

Though that the TS-monotonicity of π for each $\pi \in$ {CO,UC,TC} is apparent and has been studied in the literature [3], whether π satisfies the two variants of the TS-monotonicity criterion is not equally easy to see. In fact, we prove that among the three tournament solutions, only the top cycle satisfies the both criteria.

LEMMA 1. *TC satisfies TS-exclusive monotonicity and TS-ENM, but CO and UC do not satisfy TS-ENM.*

Due to Theorem 2 and Lemma 1, we have the following theorem.

	TC-Approval	CO-Approval	UC-Approval
CCAV	**NP-hard**	**NP-hard**	**NP-hard**
CCDV	**NP-hard**	**NP-hard**	**NP-hard**
CCAC	**NP-hard**	**NP-hard**	**NP-hard**
CCDC	**NP-hard**	**NP-hard**	**NP-hard**
DCAV	**P**	**P**	**P**
DCDV	**P**	**P**	**P**
DCAC	**NP-hard**	**NP-hard**	**NP-hard**
DCDC	**NP-hard**	**NP-hard**	**NP-hard**
CBRA	**P**	**NP-hard**	**W[2]-hard**
DBRA	**P**	**P**	**W[2]-hard**

Table 1: Complexity of control and bribery problems for π-Approval where $\pi \in$ {TC, CO, UC}. The W[2]-hardness results are with respect to the number of arcs that can be reversed in total.

THEOREM 3. *TC-Approval is monotonic, and UC-Approval and CO-Approval are not monotonic.*

Finally, we study the Pareto optimal criterion.

THEOREM 4. *TC-Approval is Pareto optimal, and CO-Approval and UC-Approval are not Pareto optimal.*

3. COMPLEXITY

The constructive/destructive multimode control problem for π-Approval is defined as follows.

Constructive/Destructive Multimode Control

Input: A set $\mathcal{C}$ of candidates, a list $\mathcal{T}$ of votes, a subset $\mathcal{D} \subseteq \mathcal{C}$, a distinguished candidate $p \in \mathcal{C} \setminus \mathcal{D}$, a sublist $\mathcal{U} \subseteq \mathcal{T}$, positive integers $k_{\text{AV}}, k_{\text{DV}}, k_{\text{AC}}, k_{\text{DC}}$.

Question: Are there $D \subseteq \mathcal{D}, C \subseteq \mathcal{C} \setminus (\mathcal{D} \cup \{p\}), V \subseteq \mathcal{T} \setminus \mathcal{U}, U \subseteq \mathcal{U}$ such that $|D| \leq k_{\text{AC}}, |C| \leq k_{\text{DC}}, |U| \leq k_{\text{AV}}, |V| \leq k_{\text{DV}}$ and p wins/loses $(A, \mathcal{F})$, where $A = ((\mathcal{C} \setminus \mathcal{D}) \setminus C) \cup D$ and $\mathcal{F} = ((\mathcal{T} \setminus \mathcal{U}) \setminus V) \cup U$?

We study 8 special cases of the above problem. Precisely, we study CCAV, CCDV, CCAC, CCDC, DCAV, DCDV, DCAC and DCDC. For $X \in$ {AV, DV, AC, DC}, CCX (DCX) is the special case of Constructive (Destructive) Multimode Control such that $k_Y = 0$ for every $Y \in$ {AV, DV, AC, DC} $\setminus \{X\}$. In addition, for $X \in$ {AV, DV, DC}, $\mathcal{D} = \emptyset$ and for $X \in$ {AC, DC, DV}, $\mathcal{U} = \emptyset$.

In addition, we study two bribery problems denoted by CBRA and DBRA. In CBRA (DBRA) we are given an election $\mathcal{E} = (\mathcal{C}, \mathcal{T})$, a distinguished candidate $p \in \mathcal{C}$, and an integer $k > 0$. The question is whether we can make p win (lose) the election by reversing at most k arcs in total in tournaments in $\mathcal{T}$. CBRA and DBRA have already been studied under the name *microbribery* [5]. However, the complexity of CBRA/DBRA for π-Approval has not been studied yet.

Our complexity results are summarized in Table 1.

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REFERENCES

[1] H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. In *AAAI*, pages 784–790, 2015.

[2] D. Baumeister, G. Erdélyi, E. Hemaspaandra, L. A. Hemaspaandra, and J. Rothe. *Computational Aspects of Approval Voting*, chapter 10, pages 199–251. Handbook on Approval Voting. Springer Berlin Heidelberg, 2010.

[3] F. Brandt, M. Brill, and P. Harrenstein. Tournament solutions. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 3. Cambridge University Press, 2016.

[4] E. Elkind and N. Shah. Electing the most probable without eliminating the irrational: Voting over intransitive domains. In *UAI*, pages 182–191, 2014.

[5] P. Faliszewski, E. Hemaspaandra, L. A. Hemaspaandra, and J. Rothe. Llull and Copeland voting computationally resist bribery and constructive control. *J. Artif. Intell. Res.*, 35:275–341, 2009.

[6] P. Faliszewski, E. Hemaspaandra, and H. Schnoor. Manipulation of Copeland elections. In *AAMAS*, pages 367–374, 2010.

[7] P. C. Fishburn and S. J. Brams. Approval voting, Condorcet's principle, and runoff elections. *Public. Choice.*, 36(1):89–114, 1981.

[8] D. M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. In D. S. Felsenthal and M. Machover, editors, *Electoral Systems*, Studies in Choice and Welfare, pages 305–326. Springer Berlin Heidelberg, 2012.

[9] A. P. Lin. The complexity of manipulating k-Approval elections. In *ICAART (2)*, pages 212–218, 2011. http://arxiv.org/abs/1005.4159.

[10] B. Monjardet. An axiomatic theory of tournament aggregation. *Math. Oper. Res.*, 3(4):334–351, 1978.

[11] C. A. Tovey S. Saarinen and J. Goldsmith. A model for intransitive preferences. In *MPREF@AAAI2014*, pages 85–89, 2014.

[12] Y. Yang. Approval voting with intransitive preferences. In *CORR*, https://arxiv.org/pdf/1702.07902.pdf, 2017.

[13] Y. Yang and J. Guo. The control complexity of r-Approval: from the single-peaked case to the general case. In *AAMAS*, pages 621–628, 2014.