

# Manipulability of Thiele Methods on Party-List Profiles

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## ABSTRACT

Recent impossibility results have shown that strategyproofness is difficult to obtain for multiwinner voting rules, especially in combination with proportionality. In this paper, we attempt to identify cases where strategyproofness can be established by considering manipulation on party-list profiles. We distinguish between three types of manipulation—subset-manipulation, superset-manipulation, and disjoint-set-manipulation. Our focus is the class of irresolute Thiele rules. For all three types of manipulation, we are able to establish that Thiele rules are strategyproof on party-list profiles for several well-known preference extensions. For superset- and disjoint-set-strategyproofness, we can extend this result to all preference extensions. We are also able to show that Thiele rules are fully strategyproof for optimistic agents on these profiles.

## KEYWORDS

Multiwinner Voting; Social Choice Theory; Strategic Manipulation

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## 1 INTRODUCTION

In multiwinner voting, agents vote on a set of candidates with the goal of electing a committee, or a subset of the candidates [14]. Applications for multiwinner voting rules range from parliamentary elections, to determining a list of nominees for an award, to online recommender systems. This paper studies strategyproofness of approval-based multiwinner voting rules [23]. In this setting, each agent is asked to provide a subset of candidates that she approves of, and a set of winning candidates is chosen based on the approvals of the agents. As with other areas of social choice theory, an important aspect of studying multiwinner voting rules is determining their susceptibility to strategic manipulation. For single-winner voting rules, a seminal impossibility result by Gibbard [19] and Satterthwaite [31] establishes the difficulty of finding strategyproof voting rules. Similar impossibility results have recently been obtained for approval-based multiwinner voting rules, demonstrating that strategyproof rules are difficult to come by if we would like them to ensure some level of proportional representation.<sup>1</sup> Peters [30] establishes that no resolute approval-based rule—one that always returns a single winning committee—can be both proportional and strategyproof, even for very weak notions

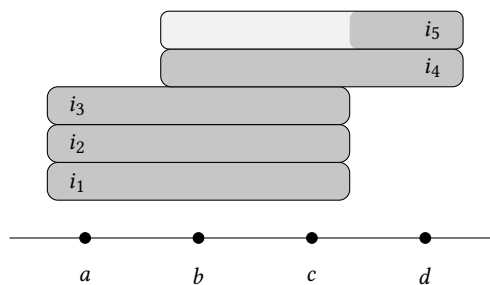
<sup>1</sup>We do not explicitly define any proportionality axioms in this paper. We refer to Aziz et al. [1] and Fernández et al. [15] for an overview of proportional representation in multiwinner voting.

of proportionality and strategyproofness. Kluiving et al. [24] show that the impossibility still remains when moving to irresolute rules. Our aim in this paper is to examine whether there are any possible “escape routes” for these impossibility results.

We study manipulation in general, but devote focus in particular to three types of manipulation: the first, subset-manipulation, is sometimes called free-riding [20, 32]. This is a simple and often successful way of manipulating multiwinner elections. Free-riding occurs when an agent omits some alternative from their set of approved candidates, and in doing so, obtains a better outcome for herself. We also study what we call superset-manipulation, and disjoint-set-manipulation, defined analogously. We note that immunity to free-riding is the strategyproofness notion used by Peters [30], meaning their impossibility result holds even for this limited type of manipulation. We give an example of free-riding using Proportional Approval Voting (PAV), a rule known to satisfy strong proportionality axioms [1]. PAV maximises the total utility of agents, where an agent’s utility for a committee containing  $m$  of her approved candidates is determined by the following formula.

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$$

EXAMPLE 1.1 (FREE RIDING). Consider the profile depicted below. Here, agents  $i_1$ ,  $i_2$ , and  $i_3$  all approve the candidates  $a$ ,  $b$  and  $c$ , while agents  $i_4$  and  $i_5$  approve candidates  $b$ ,  $c$  and  $d$ . Suppose we want to elect a committee comprising three candidates. In this profile, PAV will elect the committee  $\{a, b, c\}$  as the unique winning committee.



If the last voter  $i_5$  drops  $b$  and  $c$  from her approval set (represented in light grey) and submits the approval set  $\{d\}$ , however, the unique winner will be  $\{b, c, d\}$ —her most preferred committee. The candidates  $b$  and  $c$  have enough support without agent  $i_5$ , and dropping them from her approval set results in the inclusion of candidate  $d$  as PAV attempts to ensure all voters are represented in the outcome. Thus  $i_5$  has an incentive to manipulate in this profile by submitting a subset of her truthful approval set.  $\diamond$

Because strategyproofness results for approval multiwinner voting have largely been negative, we consider weakening requirements to identify scenarios where we can obtain positive results. We do this by considering strategyproofness on a particular type

of input—so called, party-list profiles. These are profiles where each candidate belongs to a single party, and agents approve of parties as a whole rather than any subset of the candidates. We examine whether manipulation is possible from such profiles to *any* other profile, not just those in the party-list domain. Focusing on a more well-behaved domain of profiles is one way to obtain strategyproofness. The most well-known such domain restriction is likely the single-peaked preference domain of Black [6], but many others are found in the literature [11, 13]. Our approach differs from the majority of such results as we do not stipulate that the agent manipulates to a profile within the restricted domain. Studying restricted domains has already been done for approval-based multiwinner voting. For example, Elkind and Lackner [12] define several novel restrictions for this setting, and give a polynomial time algorithm for computing the result of PAV for many of these restrictions, including the party-list domain.

Strategyproofness on party-list profiles is particularly appealing as it provides an argument for using open party-lists in multiwinner elections, rather than closed party-lists, without incurring needless risk of strategic voting. A closed-party list election only allows voters to approve a party as a whole—meaning they approve all candidates from a given party, and only those candidates. An open party-list election lets agents pick and choose candidates across all parties, thereby allowing agents to express more nuanced opinions, but also presenting them more opportunities for manipulation. Of course, we cannot know ahead of time whether the closed-list ballots are in fact expressive enough. Strategyproofness on party-list profiles guarantees that we only risk strategic manipulation under the open-list system if the closed-list system did not allow voters to express their true opinion in the first place.

Our focus in this paper is a class of multiwinner voting rules known as *Thiele methods* [21, 33].

**Related Literature.** Strategic manipulation in multiwinner elections has been studied from several angles. Lackner and Skowron [25] study strategyproofness from an axiomatic point of view, showing that Approval Voting is the only strategyproof rule that also satisfies monotonicity and independence of irrelevant alternatives, as well as the only rule satisfying their SD-strategyproofness axiom. Yang and Wang [34] study strategic aspects of multiwinner voting relative to various restrictions on the input. Laslier and Van der Straeten [26] study strategic voting of multiwinner approval voting in a probabilistic setting. Bredereck et al. [9] examine the related notion of bribery in a multiwinner setting.

On the more computational side, Bartholdi and Orlin [5] show that determining whether there exists a possible manipulation of Single Transferable Vote is NP-complete, establishing that computational complexity can be a barrier to manipulation also for multiwinner voting. More recently, the computational complexity of strategic manipulation in multiwinner voting has been studied by Aziz et al. [2], Bredereck et al. [10] and Meir et al. [27]. As an example of a more negative result, Obraztsova et al. [29] give polynomial time algorithms for manipulation of multiwinner scoring rules.

On the related notion of robustness, Bredereck et al. [8] examine how robust the outcome of multiwinner voting rules are to small changes in the input. While they interpret this as the possibility

of mistakes made by agents when submitting their preferences, we can also think of these small perturbations as strategic actions by the voters. Gawron and Faliszewski [18] study robustness in approval-based multiwinner rules, and Misra and Sonar [28] study robustness in restricted domains.

**Contribution.** We study three types of manipulation in multiwinner elections of Thiele methods. We define a class of preference extensions—functions extending agents’ preferences over committees to preferences over sets of committees—using the Gärdenfors extension as a starting point. For this class of extensions, we show that free-riding is not possible on party-list profiles when using a Thiele method. We are also able to establish a corresponding result for superset-manipulation and disjoint-set-manipulation for *all* preference extensions. For the optimistic preference extension, we show that Thiele methods are not manipulable in any manner on party-list profiles, meaning they are fully strategyproof on these profiles for optimistic agents.

**Paper Structure.** The rest of the paper is organised as follows. We first present the framework and relevant definitions in Section 2. In Section 3 we state our main result pertaining to free-riding. We finally present our results on superset-strategyproofness and disjoint-set-strategyproofness for all preference extensions in Section 4, as well as our stronger strategyproofness result for optimistic agents. We conclude in Section 5.

## 2 PRELIMINARIES

Let  $C$  be a finite set of *candidates*, and  $N = \{1, \dots, n\}$  a finite set of *agents*. A *profile*  $A = (A_1, \dots, A_n)$  is a vector of *approval sets*, where  $A_i \subseteq C$  is the set of candidates approved by agent  $i$  in the profile  $A$ . The set of *supporters*  $N_a^A$  of a candidate  $a$  in profile  $A$  is the set of agents who approve it. We write  $\mathcal{P}(C)$  to denote all subsets of  $C$ —in other words, all possible approval sets—and  $\mathcal{P}(C)^n$  to denote the set of all profiles for  $n$  agents. We write  $\mathcal{P}_k(C)$  to mean the set of all  $k$ -size subsets of  $C$ . We will often call these sets *committees*. For two profiles  $A$  and  $A'$  and an agent  $i \in N$ , we write  $A =_{-i} A'$ —and say they are  *$i$ -variants*—if  $A_j = A'_j$  for all  $j \in N \setminus \{i\}$ .

We define voting rules relative to an outcome size  $k$ . An (irresolute) *approval-based  $k$ -committee rule*  $f$  takes as input a profile  $A$  and returns a set  $f(A)$  of  $k$ -sized committees—or  *$k$ -committees*. Formally  $f$  is a function from profiles to  $k$ -sized subsets of  $C$ :

$$f : \mathcal{P}(C)^n \rightarrow 2^{\mathcal{P}_k(C)} \setminus \{\emptyset\}$$

### 2.1 Thiele Methods

The rules we will examine in this paper are all so-called *Thiele methods* [21, 33]. Given a vector of weights  $\mathbf{w} = (w_1, w_2, \dots)$ , we define the *utility*

$$u_i^A(C, \mathbf{w}) = \sum_{x=1}^{|A_i \cap C|} w_x$$

of agent  $i$  for committee  $C$ , given the approval set  $A_i$ . The  $\mathbf{w}$ -score of a committee  $C$  in a profile  $A$  is

$$u_N^A(C, \mathbf{w}) = \sum_{i \in N} u_i^A(C, \mathbf{w})$$

When the weight vector  $\mathbf{w}$  is clear from context, we will omit it from the notation and simply write  $u_i^A(C)$ . A  $k$ -committee rule

$f$  is a *Thiele method* (or Thiele rule) if for a (nonincreasing) vector of nonnegative weights  $\mathbf{w} = (w_1, w_2, \dots)$ , where  $w_i \geq w_{i+1}$ , and a profile  $\mathbf{A}$ , the rule  $f$  includes a committee  $C$  in the outcome if and only if  $C$  is a  $k$ -committee with a maximal  $\mathbf{w}$ -score in  $\mathbf{A}$ .

Thiele methods are based on the notion of diminishing returns for the agents—the second representative gained does not increase the agents’ utility as much as the first representative. Thus, tasked with deciding whether to give one agent a second representative or to give another agent their first, Thiele rules will never opt for the former over the latter.

*Proportional Approval Voting* (PAV) is the Thiele method defined by weights  $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ . PAV has been shown by Aziz et al. [1] to satisfy a strong proportional representation axiom—extended justified representation, and is therefore of particular interest.

## 2.2 Preferences and Strategyproofness

As agents submit approval sets, we need to explicitly specify their preference ranking over committees. We define agent  $i$ ’s preferences over  $k$ -committees  $\geq_i$  (with strict part  $>_i$ ) as follows: For two  $k$ -committees  $C$  and  $C'$ , it is the case that  $C \geq_i C'$  if and only if  $|A_i \cap C| \geq |A_i \cap C'|$ . We write  $C \sim_i C'$  if  $C \geq_i C'$  and  $C' \geq_i C$ —meaning agent  $i$  is indifferent between the two sets. Agents will never be confronted with committees of different sizes, so such committees are considered incomparable. Note that  $\geq_i$  is defined relative to  $A_i$ , what we take to be the agent’s truthful approval set.

**Preference Extensions.** Because the committee rules we consider are *irresolute*—meaning they sometimes return multiple winning committees—we need to account for preferences over sets of committees. We do this by defining a *preference extension*.<sup>2</sup> A preference extension  $e$  maps any given preference relation  $\geq$  over  $k$ -committees in  $\mathcal{P}_k(C)$ , to a relation  $\geq^e$  (with strict part  $>^e$ ) over sets of  $k$ -committees. For an agent  $i$  with preference relation  $\geq_i$  over committees in  $\mathbf{A}$ , we write  $\geq_i^e$  to denote her preferences over sets of committees, extended according to  $e$ . We say an agent has *e-preferences*, if  $\geq_i^e$  is her preference relation over sets of alternatives. For all preference orders we have  $X >_i^e Y$  if  $X \geq_i^e Y$  and  $Y \not\geq_i^e X$ . For preference extensions, we have the following (very weak) requirement:

- ▶  $X >_i^e Y$  implies that there exist some  $x \in X$  and  $y \in Y$  such that  $x >_i y$  and  $\{x, y\} \not\subseteq X \cap Y$ .

We highlight—and define—two well-known preference extensions, the *Optimistic extension*, which we refer to as  $o$ , and the *Gärdenfors preference extension* [17], which we refer to as  $g$ . For any  $X$  and  $Y$  in  $\mathcal{P}_k(C)$ , where  $\geq_i$  is a preference order over committees of size  $k$ , we have

$X \geq_i^o Y$  if and only if there is some  $x \in X$  s.t.  $x \geq_i y$  for all  $y \in Y$

We say an agent has optimistic preferences (or is an optimistic agent) if her preferences are extended according to the Optimistic extension. This extension has a very natural interpretation, as is implied by the name; when comparing two sets, an optimistic agent operates under the assumption that their top choice within each set will be chosen. For the Gärdenfors extension, we have  $X \geq_i^g Y$  if and only if one of the following three conditions is satisfied:

- (i)  $X \subset Y$  and  $x \geq_i y$  for all  $x \in X$  and  $y \in Y \setminus X$
- (ii)  $Y \subset X$  and  $x \geq_i y$  for all  $x \in X \setminus Y$  and  $y \in Y$
- (iii) Neither  $X \subset Y$  nor  $Y \subset X$ , and  $x \geq_i y$  for all  $x \in X \setminus Y$  and  $y \in Y \setminus X$

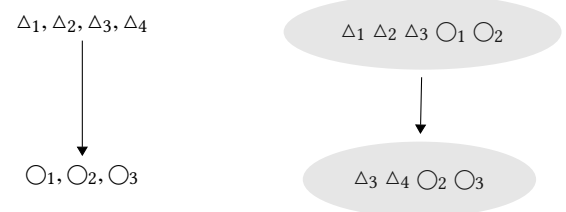
The Gärdenfors extension dictates that if one set is to be preferred over another, then new elements added should be preferred to those already in the initial set. Similarly, the elements removed should be less preferred. The elements the two sets have in common are therefore not included in any comparison, as the Gärdenfors extension only looks at how the two sets differ. We say an agent has Gärdenfors-preferences if her preferences are extended to sets of alternatives according to the Gärdenfors extension.

We now define a larger class of preference extensions which includes both the Gärdenfors and Optimistic preference extensions, as well as the Fishburn and Kelly preference extensions [16, 22].<sup>3</sup> We say a preference extension  $e$  is a *general Gärdenfors preference extension* in case that  $X >_i^e Y$  only if one of the following holds:

- (i)  $X \not\subseteq Y$  and there exists  $x \in X \setminus Y$  and  $y \in Y$  such that  $x >_i y$
- (ii)  $X \subset Y$ ,  $x \geq_i y$  for all  $x \in X$  and  $y \in Y \setminus X$ , and there exists  $x \in X$  and  $y \in Y \setminus X$  such that  $x >_i y$ .

Note that while the Gärdenfors preference extension is one specific preference extension, general Gärdenfors extensions are a class of preference extensions, of which the Gärdenfors extension is a member. We give an example below of preferences that fall into the class of general Gärdenfors preferences that are not captured by the specific extensions we have mentioned.

**EXAMPLE 2.1 (GENERAL GÄRDENFORS PREFERENCE).** Suppose we have an agent  $i$  with preferences over shapes (represented on the left), who prefers all triangles to all circles. We can define the following preference over sets of shapes: If  $X$  is a subset of  $Y$ ,  $X >_i Y$  if and only if condition (ii) above is satisfied—the agent only wants to move to a subset—thereby excluding some possibilities without adding new ones—when certain guarantees are met. Otherwise, the agent prefers the outcome with the best ratio of triangles to circles.



With such preferences, agent  $i$  would prefer a set of three triangles and two circle to one with two of each (represented on the right). This is an example of a type of preference that can be captured by the class of general Gärdenfors preferences. In this specific instance, the Gärdenfors extension would not be able to compare the two sets, as each set contains an element that is strictly preferred to an element (only) in the other;  $\Delta_4 >_i O_1$ , and  $\Delta_1 >_i O_3$ . An optimistic agent would be indifferent between the two outcomes.  $\diamond$

<sup>2</sup>For a thorough treatment of preference extensions we refer the reader to Barberà et al. [4], and for their relevance to strategyproofness we refer to Barberà [3].

<sup>3</sup>In the interest of space, we do not define the Fishburn and Kelly extensions. Their membership in the class is implied by their relation to the Gärdenfors extension (see, for example, Brandt and Brill [7]).

**Strategyproofness.** We now define the three types of manipulation we consider in this paper: subset-manipulation—or free-riding, superset-manipulation, and disjoint-set-manipulation. We then define their corresponding strategyproofness axioms.

A rule  $f$  is *manipulable* by agent  $i$  in the profile  $A$ , under preference extension  $e$ , if there exists another profile  $A' =_{-i} A$  such that  $f(A') \succ_i^e f(A)$ . We say an agent is able to *free-ride* if  $A'_i \subset A_i$ . Free-riding is particularly relevant for rules that attempt to achieve some level of representation for all voters. A free-rider omits a popular candidate from their approvals so this candidate does not count toward their representation. A rule  $f$  is *superset-manipulable* by agent  $i$  if  $A'_i \supset A_i$ , and *disjoint-set-manipulable* if  $A'_i \cap A_i = \emptyset$ . A rule  $f$  is *strategyproof* for a preference extension  $e$  if it cannot be manipulated in any profile by an agent with  $e$ -preferences. It is *immune to free-riding* for a preference extension  $e$  if no agent with  $e$ -preferences can free-ride in any profile, *superset-strategyproof* if it cannot be superset-manipulated, and *disjoint-set-strategyproofness* if it cannot be disjoint-set-manipulated.

### 2.3 The Party-List Domain

We restrict our attention in this paper to manipulation on party-list profiles. A profile  $A$  is a *party-list profile* if for all  $i, j \in N$ , either  $A_i = A_j$  or  $A_i \cap A_j = \emptyset$ . We say a rule is *strategyproof on party-list profiles*—or *party-list-strategyproof*—for a preference extension  $e$  if no agent with  $e$ -preferences can manipulate in a party-list profile.

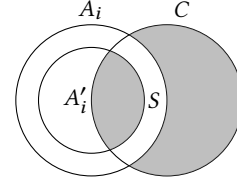
As we touched up in Section 1, party-list profiles have practical relevance for multiwinner elections. We can also show that our strategyproofness result do not hold for other domain restrictions that exist for dichotomous preferences [12], independent of the preference extension. The following example demonstrates this, suggesting that the party-list domain is indeed the most fruitful avenue to explore.<sup>4</sup>

**EXAMPLE 2.2.** Recall the profile from Example 1.1. When agent  $i_5$  submits her truthful approval set, PAV elects  $\{a, b, c\}$  as the unique winning committee. If she submits a subset of her true approval set she obtains a better outcome. Both outcomes are single committees, making the preference extension irrelevant. The initial profile in the example satisfies restrictions based on orderings of both candidates and voters such as the Candidate Extremal Interval, Voter Extremal Interval, and Weakly single-crossing requirements defined by Elkind and Lackner [12]. This of course means PAV cannot be immune to free riding on any of these restricted domains or for any weaker restriction.  $\diamond$

## 3 FREE-RIDING

Recall that free-riding describes when an agent submits a subset of their truthful approval set, in order to obtain a better outcome. We show that free-riding on party-list profiles is not possible for agents with general Gärdenfors preferences. We will also see, however, that there are quite reasonable scenarios where such manipulation, even on party-list profiles, remains possible. We first establish two Lemmas.

<sup>4</sup>We do not define these restrictions, and refer to the original paper by Elkind and Lackner [12] for a full overview.



**Figure 1: Approval sets  $A_i$  and  $A'_i$  in relation to committee  $C$ . Grey area is  $C_{\text{start}}$ . To construct  $C^*$ , add candidates from  $A'_i$  until the set reaches desired size or we run out of candidates.**

We write  $\succeq'_i$  to mean agent  $i$ 's preference order over committees, under the assumption that  $A'_i$  is the agent's truthful approval set—i.e.  $C' \succeq'_i C$  if  $|A'_i \cap C'| \geq |A'_i \cap C|$ . Lemma 3.1 identifies cases where the preference orders  $\succeq_i$  and  $\succeq'_i$  coincide.

**LEMMA 3.1.** *For profiles  $A =_{-i} A'$  and a Thiele rule  $f$  such that  $C \in f(A)$  and  $C' \in f(A')$ ,  $C' \succ_i C$  implies  $C' \succeq'_i C$ .*

**PROOF.** Let  $f$  be a Thiele rule defined by weight vector  $\mathbf{w}$ . As  $C \in f(A)$ , we know the  $\mathbf{w}$ -score of  $C$  in profile  $A$  must be at least as high as that of  $C'$ . We write this as follows, separating the utility of agent  $i$  from the utilities of agents in  $N \setminus \{i\}$ :

$$u_i^A(C) + \sum_{j \in N \setminus \{i\}} u_j^A(C) \geq u_i^A(C') + \sum_{j \in N \setminus \{i\}} u_j^A(C')$$

By assumption, we know that  $u_i^A(C) < u_i^A(C')$ , so in order for the  $\mathbf{w}$ -score of  $C$  to be at least as high as that of  $C'$ , agents in  $N \setminus \{i\}$  must—in a sense—collectively prefer  $C$  to  $C'$ :

$$\sum_{j \in N \setminus \{i\}} u_j^A(C) > \sum_{j \in N \setminus \{i\}} u_j^A(C')$$

As  $A =_{-i} A'$ , the utility of an agent in  $N \setminus \{i\}$  for any committee remains the same relative to  $A$  and  $A'$ —in other words  $u_j^A(C) = u_j^{A'}(C)$ , and  $u_j^A(C') = u_j^{A'}(C')$  for all  $j \in N \setminus \{i\}$ . Thus, we have

$$\sum_{j \in N \setminus \{i\}} u_j^{A'}(C) > \sum_{j \in N \setminus \{i\}} u_j^{A'}(C'). \quad (1)$$

Finally,  $C' \in f(A')$  implies that  $\mathbf{w}$ -score of  $C'$  in profile  $A'$  is at least as high as that of  $C$ . This, together with Equation (1) implies  $u_i^{A'}(C') > u_i^{A'}(C)$ —or  $C' \succ'_i C$ —as desired.  $\square$

We now prove a slightly more complex lemma that we will utilise multiple times throughout the paper. Lemma 3.2 establishes the existence of a particular committee  $C^*$  in the initial outcome, whenever an agent submits a subset of their initial approvals.

**LEMMA 3.2.** *Let  $f$  be a Thiele rule. For a party-list profile  $A$ , and an  $i$ -variant  $A' =_{-i} A$  where  $A'_i \subset A_i$ ,  $C \in f(A)$  implies that there is some  $k$ -committee  $C^* \in f(A)$  such that*

- $C^* \sim_i C$ , and
- for  $C' \in f(A')$  we have that  $C' \succ_i C$  implies  $(A_i \cap C^*) \subseteq A'_i$ .

**PROOF.** Let  $S = (A_i \cap C) \setminus A'_i$ . This is the set of all candidates that  $A_i$  and  $C$  agree on, except those also approved by  $A'_i$ . We will be building  $C^*$  by starting with  $C_{\text{start}} = C \setminus S$  and adding candidates

until we reach a committee of size  $k$ . See Figure 1 for a visual representation of these sets and how they relate to each other.

We add candidates to  $C_{\text{start}}$  as follows:

- ▶ If  $|A'_i \setminus C_{\text{start}}| \geq |S|$ —meaning if there are enough candidates in  $A'_i$  to fill the  $|S|$  “open spots”—we add candidates from  $A'_i \setminus C_{\text{start}}$  until we reach a committee  $C^*$  such that  $|C^*| = k$ .
- ▶ Otherwise, we add all candidates in  $A'_i$  to the committee. We then fill the remaining “open slots” with candidates from  $S$  until we reach a committee  $C^*$  of size  $k$ .

Because  $A$  is a party-list profile, it is clear from the construction of  $C^*$  that  $u_j^A(C^*) = u_j^A(C)$  for all  $j \in N$ —as  $C$  and  $C^*$  only differ on alternatives in  $A_i$ , and the two committees are of equal size. As  $C \in f(P)$ , it must therefore also be the case that  $C^* \in f(P)$ .

In order to prove the second part of the statement, suppose  $C' \in f(A')$  and  $C' \succ_i C$ . Suppose further  $|A'_i \setminus C_{\text{start}}| \not\geq |S|$ . This means we exhausted all candidates in  $A'_i$  when building  $C^*$ , and so  $A'_i \subseteq C^*$ . Because  $A'_i$  is contained in  $C^*$ , we know that  $C^* \succeq'_i C'$ . However as  $C^* \in f(A)$  and  $C' \succ_i C^*$ , Lemma 3.1 tells us that  $C' \succ'_i C^*$ , which is, of course, a contradiction.

As it cannot be the case that  $C' \succ_i C^*$  and  $|A'_i \setminus C_{\text{start}}| \not\geq |S|$ , it remains only to show that  $|A'_i \setminus C_{\text{start}}| \geq |S|$  implies  $(A \cap C^*) \subseteq A'$ . If  $|A'_i \setminus C_{\text{start}}| \geq |S|$ , then we know that all candidates added to  $C_{\text{start}}$  to create  $C^*$  must come from  $A'_i$ . In other words, we know that  $C^* \setminus C_{\text{start}} \subseteq A'_i$ —so  $A_i \cap (C^* \setminus C_{\text{start}}) \subseteq A'_i$ . Additionally, the candidates that remain in  $C_{\text{start}} \cap A_i$  are only those that are also in  $A'_i$ —so  $(C_{\text{start}} \cap A_i) \subseteq A'_i$ . Putting this together, we can see that  $(A_i \cap C^*) \subseteq A'_i$ . Thus, we have shown that  $C' \in f(A')$  and  $C' \succ_i C$  implies  $(A_i \cap C^*) \subseteq A'_i$ .  $\square$

Broadly, Proposition 3.3 establishes two things. If free-riding brings about a ‘more preferred’ committee in the manipulated outcome, then 1) that committee will already have been in the initial outcome, and 2) this committee will be accompanied by a ‘less preferred’ committee in the manipulated outcome. We take advantage of both these facts, separately, in results that will build on this one.

**PROPOSITION 3.3.** *Let  $f$  be a Thiele rule. Given an agent  $i \in N$ , profiles  $A =_{-i} A'$ —where  $A$  is a party-list profile, and approval sets  $A'_i \subset A_i$ : if  $C' \succ_i C$  for committees  $C' \in f(A')$  and  $C \in f(A)$ , then there exists a committee  $C^*$  such that  $C \sim_i C^*$ , and  $\{C^*, C'\} \subseteq f(A) \cap f(A')$ .*

**PROOF.** Let  $f$  be a Thiele rule. Suppose we have two profiles  $A$  and  $A'$ —where  $A$  is a party-list profile—and an agent  $i$  such that  $A =_{-i} A$ , and  $A'_i \subset A_i$ . Suppose further that we have committees  $C' \in f(A')$  and  $C \in f(A)$ , such that  $C' \succ_i C$ . As  $A$  is a party-list profile, Lemma 3.2 tells us there must be some  $C^* \in f(A)$  such that  $C^* \sim_i C$  and  $(A_i \cap C^*) \subseteq A'_i$ .

We first show that  $C^* \in f(A')$ . Note that  $(A_i \cap C^*) \subseteq A'_i$  and  $A'_i \subset A_i$  implies that  $|A_i \cap C^*| = |A'_i \cap C^*|$ . We also know that  $A_j = A'_j$  for all agents  $j \neq i$ . So we conclude that  $|A_j \cap C^*| = |A'_j \cap C^*|$  for all  $j \in N$ . We can express this in terms of agents’ utilities.

$$u_N^A(C^*) = u_N^{A'}(C^*) \quad (2)$$

Because  $C^* \in f(A)$ , it must hold that  $u_N^A(C^*) \geq u_N^A(C')$ . This together with Equation 2 implies that

$$u_N^A(C^*) \geq u_N^A(C')$$

Additionally, as  $A'_i \subset A_i$ , we know that  $|A_i \cap C'| \geq |A'_i \cap C'|$ . Since all other agents submit the same approval set in both profiles, we have  $u_N^A(C') \geq u_N^{A'}(C')$ , which means

$$u_N^{A'}(C^*) \geq u_N^{A'}(C')$$

As  $f$  is a Thiele rule, and  $C' \in f(A')$ , this implies that  $C^* \in f(A')$ .

We show that  $C' \in f(A)$  in a similar manner. First, since  $C' \in f(A')$ , by definition of  $f$  we know that  $u_N^{A'}(C') \geq u_N^{A'}(C^*)$ . We use Equation 2 again to conclude  $u_N^A(C') \geq u_N^A(C^*)$ . Since  $A =_{-i} A'$  and  $A'_i \subset A_i$ , we have  $u_N^A(C') \geq u_N^{A'}(C')$ , which implies  $u_N^A(C') \geq u_N^A(C^*)$ . As  $f$  is a Thiele rule and  $C^* \in f(A)$ , it must be the case that  $C' \in f(A)$ .

So we have  $\{C^*, C'\} \subseteq f(A) \cap f(A')$ , as desired.  $\square$

We will build on Proposition 3.3 in several of our strategyproofness results. Theorem 3.5 below is the first of these. We first state a Corollary that follows from Proposition 3.3 alone.

**COROLLARY 3.4.** *On party-list profiles, Thiele methods are immune to free-riding by optimistic agents.*

We are now ready to state the main result of this section.

**THEOREM 3.5.** *On party-list profiles, Thiele methods are immune to free-riding by agents with general Gärdenfors preferences.*

**PROOF.** Let  $f$  be a Thiele rule defined by weight vector  $w$ . Suppose we have a party-list profile  $A$ , and profile  $A'$  such that  $A =_{-i} A'$ , and  $A'_i \subset A_i$ . Let  $e$  be a general Gärdenfors extension. We want to show that  $f(A') \not\succeq_i^e f(A)$ . To this end, suppose we have committees  $C \in f(A)$  and  $C' \in f(A')$  such that  $C' \succ_i C$ . Proposition 3.3 then tells us that  $C' \in f(A)$ . Thus, if  $f(A') \not\subseteq f(A)$ , it cannot be the case that  $f(A') \succ_i^e f(A)$  for any general Gärdenfors preference  $e$ —as  $f(A') \succ_i^e f(A)$  implies there is some  $C' \in f(A') \setminus f(A)$  that is strictly preferred by agent  $i$  to some  $C \in f(A)$ .

So suppose  $f(A') \subset f(A)$ . From Lemma 3.2 we know there exists a committee  $C^* \in f(A)$  such that  $C^* \sim_i C$ , and  $(A_i \cap C^*) \subset A'_i$ . Note that because  $A'_i \subset A_i$  we also have that  $|A'_i \cap C^*| = |A_i \cap C^*|$ . To prove our claim, we need to consider two cases.

*Case 1:* Suppose  $A_i \subseteq C'$ —meaning  $C'$  is one of agent  $i$ ’s top choices. We show that this implies  $C' \notin f(A')$ , reaching a contradiction. We know that  $u_N^{A'}(C^*) = u_N^A(C^*)$  as  $|A'_i \cap C^*| = |A_i \cap C^*|$  and  $A =_{-i} A'$ . As  $\{C^*, C'\} \subseteq f(A)$ , we also know that  $u_N^A(C^*) = u_N^A(C')$ , so we can conclude

$$u_N^{A'}(C^*) = u_N^A(C') \quad (3)$$

Finally, as  $A_i \subseteq C'$  by assumption, we know that  $|A_i \cap C'| = |A_i|$ . As  $A'_i \subset A_i$ , this implies that  $|A'_i \cap C'| < |A_i \cap C'|$ , which, as  $A$  and  $A'$  are  $i$ -variants, implies that  $u_N^{A'}(C') < u_N^A(C')$ . Using Equation 3, we can thus conclude that  $u_N^{A'}(C') < u_N^{A'}(C^*)$ , meaning  $C' \notin f(A')$ . So we have reached a contradiction.

*Case 2:* Suppose instead that  $A_i \not\subseteq C'$ . We then need to consider two sub-cases.

- ▶ If for all  $a \in A_i \setminus A'_i$  we have  $a \in C'$ , then this means that  $A_i \setminus C' \subseteq A'_i$ . Because  $A'_i \subset A_i$ , there must exist candidates  $a \in A_i \cap C'$  such that  $a \notin A'_i$ , and  $b \in A_i \setminus C'$  such that  $b \in A'_i$ . Let  $\tilde{C} = C' \setminus \{a\} \cup \{b\}$ . Consider that for all  $j \neq i$ ,

$\tilde{C} \sim_i^e C'$ —candidates  $a$  and  $b$  are either both accepted by  $j$  or both rejected. For agent  $i$ , clearly  $\tilde{C} \succ_i^e C'$ , so we have that

$$u_N^{A'}(\tilde{C}) > u_N^{A'}(C')$$

This implies  $C' \notin f(A')$ , contradiction our initial assumption.

- Suppose instead that there exists some  $a \in A_i \setminus A'_i$  such that  $a \notin C'$ . Because Lemma 3.1 tells us that  $|A'_i \cap C'| > |A'_i \cap C|$ , we know that  $A'_i \cap C'$  is nonempty. Thus there must also exist an alternative  $b \in A'_i \cap C'$ , and clearly  $b \in A_i$ . We construct a committee  $\hat{C} = C' \setminus \{b\} \cup \{a\}$ . Because the  $\mathbf{w}$ -score of the two committees  $C'$  and  $\hat{C}$  are the same in  $A$ , and we know that  $C' \in f(A)$ , it must also be the case that  $\hat{C} \in f(A)$ . However,  $\tilde{C} \notin f(A')$ , as  $C' \sim_j^e \tilde{C}$  for all  $j \neq i$ , and  $C' \succ_i^e \tilde{C}$ , meaning  $C'$  has a strictly higher  $\mathbf{w}$ -score in  $A'$ . So  $\tilde{C} \in f(A) \setminus f(A')$ . We know from Proposition 3.3 that there is some  $C^* \in f(A')$  s.t.  $C^* \sim_i C$ . As  $\tilde{C} \sim_i C'$ , we know that  $\tilde{C} \succ_i C^*$ . Therefore it cannot be the case that  $f(A') \succ_i^e f(A)$ .

Thus we can conclude that for any general Gärdenfors preference  $e$ , we have  $f(A') \not\succeq_i^e f(A)$ .  $\square$

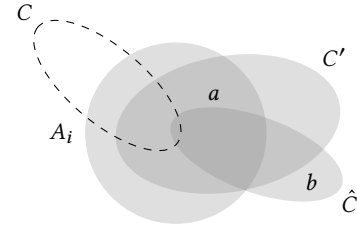
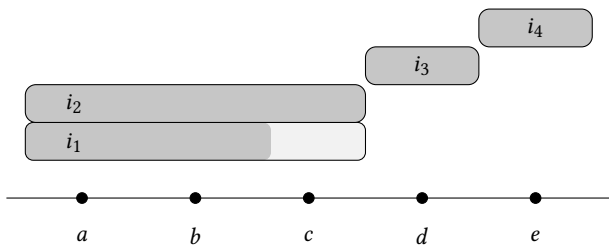
The following is an immediate consequence of Theorem 3.5, and covers three very well-known preference extensions.

**COROLLARY 3.6.** *On party-list profiles, Thiele methods are immune to free-riding for the Gärdenfors, Fishburn, and Kelly preference extensions.*

We now give an example of a specific preference extension that is not captured by our definition of general Gärdenfors preferences, and show free-riding on party-list profiles becomes possible. Our example also shows that if tie-breaking is based on a linear order that is known by the agents, then free-riding again becomes possible—meaning our results do not extend to resolute rules that are based on lexicographic tie-breaking.

**EXAMPLE 3.1 (PREFERENCES NOT COVERED BY THEOREM 3.5).** Let  $A$  be the profile depicted below where agents  $i_1$  and  $i_2$  approve of the candidates  $a, b$  and  $c$ , while agent  $i_3$  approves of only  $d$ , and agent  $i_4$  approves of only  $e$ . Clearly  $A$  is a party list profile.

Let  $k = 3$ . We use PAV to demonstrate that a manipulation is possible in this profile for agent  $i_1$ . The outcome under PAV comprises nine committees total; six with two candidates from agent  $i_1$ 's approval set:  $\{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{b, c, d\}, \{b, c, e\}$ , and three with a single candidate each from agent  $i_1$ 's approval set:  $\{a, d, e\}, \{b, d, e\}, \{c, d, e\}$ . Suppose agent  $i_1$  prefers smaller sets to larger ones, provided that for any  $C$  in the larger set, there is some  $C'$  in the smaller set such that  $C \sim_i C'$ .



**Figure 2: Representation of committees  $C, C'$  and  $\hat{C}$ , and candidates  $a$  and  $b$  used in proof of Theorem 4.1.**

Consider what happens when agent  $i_1$  drops  $c$  from their judgment set (represented in light grey). Because committees containing  $c$  will now have a lower  $\mathbf{w}$ -score, the new outcome will contain a total of four committees; two committees  $\{a, b, d\}, \{a, b, e\}$  with two of agent  $i$ 's approved candidates, and two— $\{a, d, e\}, \{b, d, e\}$ —each with a single candidate from  $A_i$ . We can see from agent  $i_1$ 's preferences that she would prefer the second (manipulated) outcome in this case. Thus  $i_1$  has an incentive to free-ride in this profile.

This example also demonstrates that our results do not hold for resolute rules that break ties according to a linear order over committees. Suppose for example that the tie-breaking order  $>$  is such that  $\{c, d, e\} > \{a, b, d\}$ , and  $\{a, b, d\} > \{a, b, e\} > \{a, d, e\} > \{b, d, e\}$ . Then the outcome in the first profile is  $\{c, d, e\}$ , while the outcome in the second is  $\{a, b, d\}$ —an improvement for agent  $i_1$ .  $\diamond$

## 4 FURTHER STRATEGYPROOFNESS RESULTS

We explore if it is possible to strengthen the main result of the preceding section. We find that on party-list profiles, we can establish superset-strategyproofness and disjoint-set-strategyproofness for all preference extensions. For the Optimistic extension, we extend the result to all types of manipulation.

### 4.1 Superset- and Disjoint-set Manipulation

We will show strategyproofness for two additional types of manipulation, superset-strategyproofness and disjoint-set-strategyproofness. This result holds for all preference extensions, including, of course, all general Gärdenfors preferences.

**THEOREM 4.1.** *On party-list profiles, Thiele methods are immune to superset-manipulation and disjoint-set-manipulation for all preference extensions.*

**PROOF.** Let  $A$  be a party-list profile, and  $f$  a Thiele rule defined by a weight vector  $\mathbf{w}$ . Suppose there is a profile  $A'$ , and committees  $C$  and  $C'$  such that  $A =_{-i} A'$ ,  $C \in f(A)$ , and  $C' \in f(A')$ . Suppose also that either  $A'_i \supset A_i$ , or  $A_i \cap A'_i = \emptyset$ . We want to show that  $C' \succ_i C$  implies  $\{C, C'\} \subseteq f(A) \cap f(A')$ . With this goal in mind, suppose  $C' \succ_i C$ .

We first show that  $C' \in f(A)$ . Because  $A$  is a party-list profile and  $C' \succ_i C$ , there must be some  $\hat{C} \in f(A)$  such that  $\hat{C} \sim_i C$  and  $(A_i \cap C') \supset (A_i \cap \hat{C})$ , where  $C$  and  $\hat{C}$  only differ on alternatives in  $A_i$ . To see why this is the case, note that the differences among  $C$  and  $\hat{C}$  pertain only to candidates in  $A_i$ , and as such  $\hat{C}$  will have the same  $\mathbf{w}$ -score as  $C$  in  $A$ . We use the fact that  $\hat{C} \in f(A)$  to show

$C' \in f(\mathbf{A})$ . For a visual representation of these committees and the candidates we will reference, see Figure 2.

Let  $a \in A_i$  be an alternative such that  $a \in C' \setminus \hat{C}$ . Such an alternative must exist as  $C' \succ_i \hat{C}$ . Because  $\mathbf{A}$  is a party-list profile, and  $|C'| = |\hat{C}|$ , we know there must be some party with (strictly) fewer representatives in  $C'$  than in  $\hat{C}$ . In other words, there exists some alternative  $b \notin A_i$  such that  $b \in \hat{C} \setminus C'$  and  $|A_j \cap \hat{C}| > |A_j \cap C'|$  for all  $j \in N_b^{\mathbf{A}}$ . We define a  $k$ -committee  $C_1 = \hat{C} \setminus \{b\} \cup \{a\}$ .

Our immediate goal is to show  $C_1 \in f(\mathbf{A})$ . Note that the  $w$ -score of  $C_1$  in  $\mathbf{A}$  cannot be higher than that of  $\hat{C}$ , as this would imply  $\hat{C} \notin f(\mathbf{A})$ . Because the two committees differ only with regard to alternatives  $a$  and  $b$ , we can express this as follows:

$$\sum_{j \in N_b^{\mathbf{A}}} w_{|A_j \cap \hat{C}|} \geq \sum_{j \in N_a^{\mathbf{A}}} w_{|A_j \cap \hat{C}|+1}$$

Similarly, the committee  $C' \setminus \{a\} \cup \{b\}$  cannot have a higher  $w$ -score than  $C'$  in  $\mathbf{A}'$ , so it must hold that

$$\sum_{j \in N_a^{\mathbf{A}'}} w_{|A'_j \cap C'|} \geq \sum_{j \in N_b^{\mathbf{A}'}} w_{|A'_j \cap C'|+1}$$

We want to connect the two inequalities above. We know that  $|A_j \cap C'| > |A_j \cap \hat{C}|$  for all  $j \in N_a^{\mathbf{A}}$ . For all  $j \neq i$  this immediately tells us  $|A'_j \cap C'| > |A_j \cap \hat{C}|$ , as  $\mathbf{A} =_{-i} \mathbf{A}'$ . For agent  $i$ , we know either  $A'_i \supset A_i$ , or  $A'_i \cap A_i = \emptyset$ . If  $a \in A'_i$ , then it must be that  $A'_i \supset A_i$ , meaning  $|A'_i \cap \hat{C}| \geq |A_i \cap \hat{C}|$ . From Lemma 3.1 we know that  $C' \succ_i \hat{C}$ ,  $\hat{C} \in f(\mathbf{A})$ , and  $C' \in f(\mathbf{A}')$  implies  $|A'_i \cap C'| > |A'_i \cap \hat{C}|$ , so we can conclude that  $|A'_j \cap C'| > |A_j \cap \hat{C}|$  for all  $j \in N_a^{\mathbf{A}'}$ . Because  $w$  is a non-increasing weight vector, this implies  $w_{|A'_j \cap C'|} \leq w_{|A_j \cap \hat{C}|+1}$  for all  $j \in N_a^{\mathbf{A}'}$ . Given that  $N_a^{\mathbf{A}} \supseteq N_a^{\mathbf{A}'}$ , we can conclude that

$$\sum_{j \in N_a^{\mathbf{A}}} w_{|A_j \cap \hat{C}|+1} \geq \sum_{j \in N_a^{\mathbf{A}'}} w_{|A'_j \cap C'|}$$

We can now build the following chain of inequalities:

$$\begin{aligned} \sum_{j \in N_b^{\mathbf{A}}} w_{|A_j \cap \hat{C}|} &\geq \sum_{j \in N_a^{\mathbf{A}}} w_{|A_j \cap \hat{C}|+1} \\ &\geq \sum_{j \in N_a^{\mathbf{A}'}} w_{|A'_j \cap C'|} \\ &\geq \sum_{j \in N_b^{\mathbf{A}'}} w_{|A'_j \cap C'|+1} \end{aligned} \quad (4)$$

Recall that for all  $j \in N_b^{\mathbf{A}}$ , it is the case that  $|A_j \cap \hat{C}| > |A_j \cap C'|$ , which—as  $A_j = A'_j$ —is equivalent to  $|A_j \cap \hat{C}| > |A'_j \cap C'|$ . This implies  $w_{|A_j \cap \hat{C}|} \leq w_{|A'_j \cap C'|+1}$ . As  $N_b^{\mathbf{A}} \subseteq N_b^{\mathbf{A}'}$ , we then have that

$$\sum_{j \in N_b^{\mathbf{A}}} w_{|A_j \cap \hat{C}|} \leq \sum_{j \in N_b^{\mathbf{A}'}} w_{|A'_j \cap C'|+1} \quad (5)$$

Equations 4 and 5 together imply that our chain of inequalities “collapses”, meaning we get

$$\sum_{j \in N_b^{\mathbf{A}}} w_{|A_j \cap \hat{C}|} = \sum_{j \in N_a^{\mathbf{A}}} w_{|A_j \cap \hat{C}|+1}$$

This can only be the case if  $C_1 \in f(\mathbf{A})$ .

Finally, to see that  $C_1 \in f(\mathbf{A})$  implies that  $C' \in f(\mathbf{A})$ , consider the following. We know that  $C_1$  is created by adding one candidate from  $C'$  and removing one candidate that is not in  $C'$ . If  $|\hat{C} \setminus C'| = 1$ , then  $C_1 = C'$  and we are done. Otherwise, note that  $|C_1 \setminus C'| = |\hat{C} \setminus C'| - 1$ , meaning  $C_1$  is one candidate closer to  $C'$  than  $\hat{C}$ . Importantly, we also know the following:

- (i)  $C' \succ_i C_1$ ,
- (ii)  $(A_i \cap C') \supset (A_i \cap C_1)$ , and
- (iii)  $C_1 \in f(\mathbf{A})$ .

Thus, we can use the same argument we used to show that  $C_1 \in f(\mathbf{A})$  to show there is some committee  $C_2 \in f(\mathbf{A})$  such that  $|C_2 \setminus C'| = |\hat{C} \setminus C'| - 2$ . We can repeat this argument until we reach a committee  $C_m \in f(\mathbf{A})$  where  $m = |\hat{C} \setminus C'|$ , meaning  $C_m = C'$ .

We now show that  $C \in f(\mathbf{A}')$ . We omit some details as the argument proceeds in a similar fashion. Note that as  $C' \in f(\mathbf{A})$ , and  $\mathbf{A}$  is a party-list profile, we know there must be some  $\hat{C}' \in f(\mathbf{A})$  such that  $|A_j \cap C'| = |A_j \cap \hat{C}'|$  for all  $j \in N$ , and  $A_i \cap \hat{C}' \supset A_i \cap C$ , where  $C'$  and  $\hat{C}'$  only differ on alternatives in  $A_i$ . This is because  $C' \in f(\mathbf{A})$ , and  $\hat{C}'$  will have the same  $w$ -score as  $C'$  in  $\mathbf{A}$ . Because  $C'$  and  $\hat{C}'$  only differ on alternatives in  $A_i$ , and both are  $k$ -sized committees, it must be the case that  $|A'_i \cap \hat{C}'| = |A'_i \cap C'|$ . As  $\mathbf{A} = \mathbf{A}'$ , we know  $|A'_j \cap \hat{C}'| = |A_j \cap C'|$  for all  $j \neq i$  as well, so we can conclude that  $\hat{C}' \in f(\mathbf{A}')$ —as it would have the same  $w$ -score as  $C'$  in  $\mathbf{A}'$ .

We repeat a similar argument as we did when showing  $C' \in f(\mathbf{A})$ . We have some  $a' \in A_i$  such that  $\hat{C}' \setminus C$ , and some  $b' \in C \setminus \hat{C}'$  such that for all  $j \in N_b^{\mathbf{A}'}$  we have  $|A_j \cap C| > |A_j \cap \hat{C}'|$ . Let  $C'_1 = C' \setminus \{b\} \cup \{a\}$ . Arguing in almost exactly the same way as above, we show that

$$\sum_{j \in N_b^{\mathbf{A}'}} w_{|A'_j \cap C'|+1} = \sum_{j \in N_a^{\mathbf{A}'}} w_{|A'_j \cap C'|}$$

Thus  $\hat{C}'_1 \in f(\mathbf{A}')$ . We can again repeat this argument to show that  $C \in f(\mathbf{A}')$ .

So we have shown that for any  $C' \in f(\mathbf{A}')$  and  $C \in f(\mathbf{A})$ , if  $C' \succ_i C$ , then  $\{C, C'\} \subseteq f(\mathbf{A}) \cap f(\mathbf{A}')$ , meaning it cannot be the case that  $f(\mathbf{A}') \succ_i^e f(\mathbf{A})$ .  $\square$

Corollary 4.2 follows from Theorem 3.5 and Theorem 4.1.

**COROLLARY 4.2.** *On party-list profiles, Thiele methods are superset-strategyproof, disjoint-set-strategyproof, and immune to free-riding for general Gärdenfors preferences.*

Our results paint a positive picture for Thiele rules on party-list profiles as they rule out all three types of manipulation considered in this paper for a large class of preferences. Importantly, we are also able to establish some level of strategyproofness for these rules that does not depend on the preference extension.

## 4.2 Optimistic Agents

We obtain our strongest result for the Optimistic preference extension, as we can establish full strategyproofness for Thiele methods on party-list profiles.

We will be working with profiles which are not party-list profiles, but are one agent away from a party-list profile. We write  $\mathbf{A}_{-i}$  to mean the profile  $\mathbf{A}$  with the approval set of agent  $i$  removed. An agent  $i$  casts a *separable vote* in a profile  $P$  if for all agents  $j \in N$  either  $A_i \subseteq A_j$  or  $A_i \cap A_j = \emptyset$ .

We now show that optimistic agents have no incentive to superset-manipulate. We will use this to establish our strategyproofness result for optimistic agents.

LEMMA 4.3. *Let  $f$  be a Thiele rule, and  $A$  a profile such that  $A_{-i}$  is a party-list profile, and  $A_i$  is a separable vote in  $A$ . Given an agent  $i \in N$ , and a profile  $A =_{-i} A'$  such that  $A'_i \supset A_i$ : if  $C' >_i C$  for all  $C \in f(A)$ , then  $C' \notin f(A')$ .*

PROOF. Let  $f$  be a Thiele rule defined by weight vector  $w$ . Suppose we have  $A$  and  $A'$ —where  $A_{-i}$  is a party-list profile, and  $A_i$  is a separable vote in  $A$ —and an agent  $i \in N$  such that  $A =_{-i} A'$ , and  $A'_i \supset A_i$ . Let  $C \in f(A)$  be among the most preferred committees for agent  $i$  in  $f(A)$ , and suppose there exists a  $k$ -committee  $C'$ , such that  $C' >_i C$ . We want to show that  $C' \notin f(A')$ .

We identify two candidates relevant for our purposes. Because  $C' >_i C$ , we know there must exist at least one candidate  $a \in A_i \cap (C' \setminus C)$ . We know that there is at least one party with strictly fewer representatives in  $C'$  than in  $C$ , as they are both committees of the same size. More formally, because  $A_{-i}$  is a party-list profile, there must also exist some candidate  $b \in C \setminus (C' \cap A_i)$ , such that for all  $j \in N_b^A$ , it is the case that  $|A_j \cap C| > |A_j \cap C'|$ . If this were not the case, then  $C' \geq_j C$  for all  $j \neq i$  and  $C' >_i C$ , meaning  $C \notin f(A)$ , contradicting our initial assumption.

First, we want to show that  $A_j \cap C = A_i \cap C$  for all  $j \in N_a^A$ . We know that  $A_i \subseteq A_j$  for all  $j \in N_a^A$  (and  $A_j = A_{j'}$  for  $j, j' \in N_a^A \setminus \{i\}$ ). Recall that  $a \notin C$ . Suppose  $(A_j \setminus A_i) \cap C \neq \emptyset$ , meaning there is some alternative  $x \in A_j \setminus A_i$  such that  $x \in C$ . Then the committee  $C \setminus \{x\} \cup \{a\}$  will have a  $w$ -score higher than  $C$  in the profile  $A$ . As this implies  $C \notin f(A)$ , we can conclude that  $(A_j \setminus A_i) \cap C = \emptyset$ . In other words, we have that  $A_j \cap C = A_i \cap C$  for all  $j \in N_a^A$ . As  $A_i \subseteq A_j$  for all  $j \in N_a^A$ , we know  $|A_j \cap C'| \geq |A_i \cap C'|$ . As  $|A_i \cap C'| > |A_i \cap C|$  by assumption, this implies that  $|A_j \cap C'| > |A_j \cap C|$ . As  $w$  is a non-increasing weight vector, we this implies that  $w_{|A_j \cap C'|} \leq w_{|A_j \cap C|+1}$  for all  $j \in N_a^A$ . Similarly we know  $|A_j \cap C| > |A_j \cap C'|$  for all  $j \in N_b^A$ . Thus we have

$$\begin{aligned} \sum_{j \in N_b^A} w_{|A_j \cap C|+1} &\geq \sum_{j \in N_b^A} w_{|A_j \cap C|} \\ \sum_{j \in N_a^A} w_{|A_j \cap C|+1} &\geq \sum_{j \in N_a^A} w_{|A_j \cap C'|} \end{aligned}$$

We also claim the following:

$$\sum_{j \in N_b^A} w_{|A_j \cap C|} > \sum_{j \in N_a^A} w_{|A_j \cap C|+1}$$

To see why this is the case, note that if it did not hold, then the committee  $(C \setminus \{b\}) \cup \{a\}$  would have a  $w$ -score at least as high as  $C$ , and thus would be among the winning committees in  $f(A)$ . This would clearly contradict our assumption that  $C \in f(A)$  is one of the most preferred outcomes for agent  $i$  in  $f(A)$ , as  $(C \setminus \{b\}) \cup \{a\} >_i C$ . Putting together the above, we conclude that

$$\sum_{j \in N_b^A} w_{|A_j \cap C|+1} > \sum_{j \in N_a^A} w_{|A_j \cap C'|} \quad (6)$$

We can now show  $C' \notin f(A')$ . Let  $\tilde{C} = (C' \setminus \{a\}) \cup \{b\}$  be a  $k$ -committee. We calculate the  $w$ -score of  $\tilde{C}$  in  $A'$ .

$$u_N^{A'}(\tilde{C}) = u_N^{A'}(C') - \sum_{j \in N_a^A} w_{|A_j \cap C'|} + \sum_{j \in N_b^A} w_{|A_j \cap C|+1}$$

Taken together with Equation 6, the above implies that  $\tilde{C}$  has a  $w$ -score strictly higher than  $C'$  in  $A'$ , meaning  $C' \notin f(A')$ .  $\square$

We can now use Proposition 3.3, which speaks about free-riding, and Lemma 4.3, which pertains to superset-manipulation, to prove the following Theorem for optimistic agents.

THEOREM 4.4. *Thiele methods are party-list-strategyproof for optimistic agents.*

PROOF. Let  $A$  be a party-list profile, and let  $A'$  be a profile such that  $A =_{-i} A'$ . Suppose there exists some  $C'$  such that  $C' >_i C$  for any  $C \in f(A)$ . Let  $f$  be a Thiele rule. We show that  $C' \notin f(A')$ .

Suppose for contradiction that  $C' \in f(A')$ . We construct a third, intermediate, profile  $A^*$  where  $A^* =_{-i} A$  and  $A_i^* = A_i \cap A'_i$ . Note that  $A^*_{-i}$  is a party-list profile, and  $A_i^*$  is a separable vote in  $A$ . We assume  $A_i^* \neq A_i$ —if this were not the case, then  $f(A) = f(A^*)$ , and Lemma 4.3 alone would be enough to establish our claim.

As  $C' \in f(A')$  and  $A'_i \supset A_i^*$ , we know from Lemma 4.3 that there exists some  $C^* \in f(A^*)$  such that  $C^* \geq_i C'$ . If this were not the case, then  $C'$  would be strictly preferred to all committees in  $f(A^*)$ , and so  $C' \notin f(A)$ , which would be a contradiction. As  $C^* \geq_i C'$ , we know that  $C^* >_i C$ . Because  $A_i^* \subset A_i$ , we can use Proposition 3.3 to show that  $C^* \in f(A^*)$  implies  $C^* \in f(A)$ . This contradicts our assumption that  $C' >_i C$  for all  $C \in f(A)$  as  $C^* \geq_i C'$ .  $\square$

Note that Theorem 4.4 makes no assumptions about how agents may manipulate, as any possible manipulation amounts to an agent first removing some candidates from their approval set (possibly none), and subsequently adding new candidates (again, possibly none). As the optimistic preference extension is a very natural and intuitive extension, Theorem 4.4 is a very welcome result.

## 5 CONCLUSION

We have studied strategyproofness of Thiele methods on party-list profiles. In particular, we focused on three types of manipulation: free-riding, superset-manipulation, and disjoint-set manipulation. We defined a class of preference extensions, the general Gärdenfors preferences, and obtained positive results for all three types of manipulation on party-list profiles. We showed for general Gärdenfors preferences that it is not possible to manipulate Thiele rules in any of these three manners on party-list profiles. For superset- and disjoint-set-manipulation, we have shown this for all preference extensions. We have also shown that Thiele methods are fully strategyproof on party-list profiles for optimistic agents.

Strategyproofness on party-list profiles spells good news for many applications of multiwinner voting. Our results also suggest that focusing on particular domains or profiles may be a fruitful avenue of study for establishing further strategyproofness results. We have only studied approval-based rules, but similar methods may also yield strategyproofness results for multiwinner rules that aggregate preference rankings.



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