On the Relative Succinctness of Modal Logics with Union, Intersection and Quantification

Wiebe van der Hoek
University of Liverpool
Liverpool, UK
wiebe@liverpool.ac.uk

Petar Iliev
LORIA, CNRS - University of Lorraine
Nancy, France
petar.iliev@loria.fr

ABSTRACT

In the study of knowledge representation formalisms, there is a current interest in the question of how different formal languages compare in their ability to compactly express semantic properties. Recently, French *et al.* [9] have shown that modal logics with a modality for public announcement, for everybody knows, and for somebody knows are all exponentially more succinct than basic modal logic. In this paper we compare the above mentioned logics not with basic modal logic but with each other and also with modal logics that have a modality for distributed knowledge. Interestingly, modal logic with such a modality is more expressive than the other modal logics mentioned, but still we can show that some of those weaker logics are exponentially more succinct than the former. Additionally, we prove that the opposite is also possible: indeed, we show that modal logic with a modality for distributed knowledge is more succinct than modal logic with a modality for everybody knows.

Categories and Subject Descriptors

I.2 [**Artificial Intelligence**]: Knowledge Representation Formalisms and Methods

Keywords

Knowledge Representation, Modal Logic, Succinctness

1. INTRODUCTION

The great number of logics for agent attitudes and behaviour in formal approaches to agency [18], and, among others, several description logics [4] in knowledge representation, dynamic logics [14] in computer science and, generally, a zoo of modal logics [5] for artificial intelligence, leads us to the natural question of how to actually compare those logics. Of course some well-known criteria are expressivity, decidability and computational complexity, but, to summarise an argument given in [7] it is quite often the case that logics are equally expressive, and either have similar computational complexity properties, or their respective complexities are so high that the difference is almost meaningless in practical situations. Hence, [7, 6] suggest another potentially very important criterion, namely *representational succinctness.* Intuitively, if we are interested in some particular semantic property Q that is expressible with formulae φ_1 and φ_2 from two formalisms L_1 and L_2 respectively, we can ask if there is a significant difference in the lengths of φ_1 and φ_2, i.e. whether one of them is more succinct than the other. In this sense, the notion of succinctness is a refinement of the notion of expressivity.

Of course, rather than comparing the lengths of just two formulae that express a single property, we will be actually interested in showing (see for example [9, Lemma 1]) that there is an infinite sequence of properties $Q_1, Q_2, \dots$ of some class of models $\mathbb{M}$ such that for every n there is a φ_n in L_1 that expresses Q_n, in such a way that the lengths of the φ_n's grow polynomially in n, while the length of any formula ψ_n from L_2 that express Q_n is at least 2^n. If this is true, we say that L_1 is exponentially more succinct than L_2 on $\mathbb{M}$ and write $L_1 \preceq^{EXP}_{\mathbb{M}} L_2$.

The starting point of our investigations is modal logic, denoted $[i]$ML, where, for each i in some index set I, the formula $[i]\varphi$ is true in a point[1] s if all points that are reachable from s in one step along the relation R_i satisfy φ. $[i]$ML has many applications, so 'reachability' here can refer to epistemic indistinguishability for an agent i, a transition caused by program i, or, in description logic, a role, like 'has as a friend'. In its abstract form, R_i is nothing more than a set of pairs, and Boolean modal logic ([10]) generalises this further by introducing formulae $[\cup_\Gamma]\varphi$ for every $\Gamma \subseteq I$ that simply say "φ is true in all t such that $s(\bigcup_{i\in\Gamma} R_i)t$". One can easily show that this is the same as requiring that for all $i \in \Gamma$ the formula $[i]\varphi$ is true in s. That is why this operator is sometimes written as $[\forall_\Gamma]$. In description logic, $[\cup_\Gamma]$ denotes role disjunction, e.g., one can refer to all siblings as a union of one's brothers and sisters. In dynamic logic, $[\cup_\Gamma]\varphi$ denotes demonic non-determinism: no matter which program from the set of programs Γ is executed at s the resulting state t will satisfy φ. In epistemic logic, $[\cup_\Gamma]\varphi$ means "everybody in the group of agents Γ knows that φ is true". We use $[\cup]ML$ to denote ML extended with formulae $[\cup_\Gamma]\varphi$.

Analogously, we can introduce formulae $[\exists_\Gamma]\varphi$ to mean "somebody in the group of agents Γ knows φ" by stipulating that $[\exists_\Gamma]\varphi$ is true at a point s when there is an $i \in \Gamma$ such that $[i]\varphi$ is true at s. Indeed, this notion has natural counterparts in description logic (denoting the individuals of whom either all friends or all foes are male) and dynamic logic (angelic non-determinism: there is a program in Γ,

[1]We use the term point as a general term for 'state', 'world', 'object', etc.

Appears in: *Alessio Lomuscio, Paul Scerri, Ana Bazzan, and Michael Huhns (eds.), Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2014), May 5-9, 2014, Paris, France.*

that, when executed at the current point, will lead to a point that satisfies φ). *ML* extended with formulae $[\exists_\Gamma]\varphi$ will be denoted $[\exists]ML$. Note that $[\cup_\Gamma]\varphi$ and $[\forall_\Gamma]\varphi$ are the same (something holds for arbitrary Γ-successors iff it holds for each of them). However, as we will shortly see (Definition 2) $[\cap_\Gamma]\varphi$ and $\exists_\Gamma\varphi$ are different (for something to hold at a common Γ-successor is different from it holding in all i-successors, for some $i \in \Gamma$).

Another natural operator in Boolean modal logic is the intersection modality $[\cap_\Gamma]$. Intuitively, the formula $[\cap_\Gamma]\varphi$ is true in a point s when all points t such that $s(\bigcap_{i\in\Gamma} R_i)t$ satisfy φ. In epistemic logic, this yields the notion of distributed knowledge, in dynamic logic $[\cap_\Gamma]$ formalises the notion of parallel execution of the programs in Γ, and in description logic it denotes role conjunction, allowing one to express for instance disjointness of two roles R and S, as in description logic notation $\top \sqsubseteq \forall(R \sqcap S).\bot$. Let $[\cap_\Gamma]ML$ denote ML extended with formulae $[\cap_\Gamma]\varphi$

A final construct we will study in this paper is that of domain restriction, which is achieved by formulae of the form $[\varphi]\psi$. Intuitively, such a formula is true at a point s if after removing all points that do not satisfy the formula φ, the formula ψ is true at s in the resulting new model. This operator is called public announcement in (dynamic) epistemic logic, but in description logic it would allow for some hypothetical reasoning: "assuming that all objects satisfying $\neg\varphi$ are removed from the domain, then ψ would be true about the current object". We denote *ML* extended with formulae $[\varphi]\psi$ by $[\varphi]ML$.

Although $[\varphi]ML$ and ML are equally expressive and the computational complexity of their satisfiability problems is the same, Lutz showed in [15] that $[\varphi]ML$ is exponentially more succinct than *ML* on the class of all Kripke models. French et al. [9] strengthened this result to showing that it is also true on the class of $\mathbb{S}_5$ models, the models typically used in epistemic logic. It was also shown in [9] that both $[\cup]ML$ and $[\exists_\Gamma]ML$ are exponentially more succinct than *ML*.

In this paper, we compare in terms of succinctness the logics $[\cup]ML$, $[\exists]ML$, $[\cap]ML$ and $[\varphi]ML$. We would like to stress that $[\cup]ML$, $[\exists]ML$, and $[\varphi]ML$ are equally expressive whereas $[\cap]ML$ is more expressive than these three logics. Nevertheless, some of our results show that there are Kripke models on which $[\cup]ML$, $[\exists]ML$, and $[\varphi]ML$ are exponentially more succinct than $[\cap]ML$.

The rest of this paper is organised as follows. In Section 2 we formally define the logics under consideration, we give a working definition of succinctness and present Formula Size Games, which will be the main tool we use in the proofs of our results. In Section 3 we briefly discuss some existing results in this area after which we present our main contribution. Section 4 shows how there results are obtained, and in Section 5, we round off.

2. PRELIMINARIES

Let a finite index set $I = \{i_1, i_2, \ldots, i_m\}$ and a countable set of propositional symbols $P = \{p_1, p_2, \ldots\}$ be given. From now on, we will use p to denote an arbitrary element of P, i to denote an element of I, and Γ to denote an arbitrary subset of I that contains at least two indices.

Modal languages

Definition 1. *A modal language $\Phi_{[X]}$ is a set of formulae over P and I and a parameter $\Box \in \{[i], [\cap_\Gamma], [\cup_\Gamma], [\exists_\Gamma], [\varphi]\}$ generated by the following BNF*

$$\varphi \;:=\; p \;\mid \neg\varphi \mid \varphi \vee \varphi \mid \Box\varphi.$$

Depending on our choice for $\Box$, we obtain the languages:

- $\Phi_{[i]}$*:* $\Box \in \{[i] \mid i \in I\}$
- $\Phi_{[\cap]}$*:* $\Box \in \{[i], [\cap_\Gamma] \mid i \in I, \Gamma \subseteq I\}$
- $\Phi_{[\cup]}$*:* $\Box \in \{[i], [\cup_\Gamma] \mid i \in I, \Gamma \subseteq I\}$
- $\Phi_{[\exists]}$*:* $\Box \in \{[i], [\exists_\Gamma] \mid i \in I, \Gamma \subseteq I\}$
- $\Phi_{[\cup,\cap]}$*:* $\Box \in \{[i], [\cup_\Gamma], [\cap_\Gamma] \mid i \in I, \Gamma \subseteq I\}$
- $\Phi_{[\varphi]}$*:* $\Box \in \{[i], [\varphi] \mid i \in I, \varphi \in \Phi_{[\varphi]ML}\}$

The length $|\varphi|$ of a formula φ is defined as follows: $|p| = 1, |\neg\varphi| = 1 + |\varphi|$ and $|\varphi \vee \psi| \;=\; |[\varphi]\psi| = |\varphi| + |\psi|$. For the other boxes, we stipulate $|\Box\psi| = 1 + |\psi|$. We would like to stress that we could have taken into consideration the size $|\Gamma|$ when defining the size of a box-formula that uses Γ but this would not have changed our succinctness results. For any $\Box$, we write $\Box^n\varphi$ to mean the sequence consisting of n boxes, followed by φ.

A *logic* is a tuple $[X]ML = \langle \Phi_{[X]}, \models, \mathbb{M}\rangle$, where $\Phi_{[X]}$ is a language, $\mathbb{M}$ is a class of models and $\models$ is a binary relation, also known as truth definition, between a model from $\mathbb{M}$ and a formula $\varphi \in \Phi_{[X]}$. So we talk about the logic $[i]ML$, the logic $[\varphi]ML$, the logic $[\cup,\cap]ML$, etc.

Definition 2. *Given I and P, a Kripke model is a triple $\mathscr{M} = \langle M, R, V\rangle$, where M is a non-empty set, $R : I \to 2^{M\times M}$ is a mapping that assigns to every index i a relation R_i on M, and $V : P \to 2^M$ is a function that gives for every $a \in M$ the atoms $V(a)$ that are true there. The truth of a formula is defined in a pointed model $(\mathscr{M}, s)$, where $s \in M$.*

$(\mathscr{M}, s) \models p$	*iff*	$s \in V(p)$;
$(\mathscr{M}, s) \models \neg\psi$	*iff*	*not* $(\mathscr{M}, s) \models \psi$;
$(\mathscr{M}, s) \models \varphi \vee \psi$	*iff*	$(\mathscr{M}, s) \models \varphi$ *or* $(\mathscr{M}, s) \models \psi$;
$(\mathscr{M}, s) \models [i]\psi$	*iff*	*for all* t *with* $sR_it, (\mathscr{M}, t) \models \psi$;
$(\mathscr{M}, s) \models [\exists_\Gamma]\psi$	*iff*	*for some* $i \in \Gamma, (\mathscr{M}, s) \models [i]\psi$;
$(\mathscr{M}, s) \models [\cup_\Gamma]\psi$	*iff*	*for all* t *with* $s \cup_{\gamma\in\Gamma} R_\gamma t, (\mathscr{M}, t) \models \psi$;
$(\mathscr{M}, s) \models [\cap_\Gamma]\psi$	*iff*	*for all* t *with* $s \cap_{\gamma\in\Gamma} R_\gamma t, (\mathscr{M}, t) \models \psi$;
$(\mathscr{M}, s) \models [\varphi]\psi$	*iff*	*If* $(\mathscr{M}, s) \models \varphi$, *then* $(\mathscr{M}|_\varphi, s) \models \psi$.

Intuitively, the model $\mathscr{M}|_\varphi$ used to define the $\models$ relation for the formula $[\varphi]\psi$ is the restriction of the model $\mathscr{M}$ to the points in which φ is true. Formally, for any formula φ, and any model $\mathscr{M} = \langle M, R, V\rangle$, the model $\mathscr{M}|_\varphi = \langle M', R', V'\rangle$, is such that $M' = \{v \in M | (\mathscr{M}, v) \models \varphi\}$, and R' and V' are the restrictions of R and V to M'. We will also use the dual $\langle\varphi\rangle$ of $[\varphi]$, where $\langle\varphi\rangle\psi$ is defined as $\neg[\varphi]\neg\psi$: in other words, $(\mathscr{M}, s) \models \langle\varphi\rangle\psi$ if and only if $(\mathscr{M}, s) \models \varphi$ and $(\mathscr{M}|_\varphi, s) \models \psi$.

If $\mathbb{A}$ is a set of pointed models and φ is a formula of one of the logics above, we write $\mathbb{A} \models \varphi$ to mean that *for all* $(\mathscr{M}, s) \in \mathbb{A}$, $(\mathscr{M}, s) \models \varphi$. Two classes of models are of particular interest in this paper: $\mathbb{K}$ (the class of all Kripke models) and $\mathbb{S}5$ (the class of epistemic models, where each R_i is an equivalence relation).

Succinctness.

The definition of expressivity in logic is standard: Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M}\rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M}\rangle$ be two logics. We say that L_2 is at least as expressive as L_1 on the class of models $\mathbb{M}$, and write $L_1 \leq_{\mathbb{M}} L_2$, if and only if for every formula $\varphi_1 \in \Phi_1$, there is a formula $\varphi_2 \in \Phi_2$ such that for every $(\mathscr{M}, w) \in \mathbb{M}$, it is true that $(\mathscr{M}, w) \models_1 \varphi_1$ if and only if $(\mathscr{M}, w) \models_2 \varphi_2$. We say in such a case that the formula φ_2 is equivalent to φ_1 *on* $\mathbb{M}$, and write $\varphi_1 \equiv_{\mathbb{M}} \varphi_2$.

L_1 and L_2 are said to be equally expressive on $\mathbb{M}$, written $L_1 =_{\mathbb{M}} L_2$, if both $L_1 \leq_{\mathbb{M}} L_2$ and $L_2 \leq_{\mathbb{M}} L_1$ hold. Finally, $L_1 <_{\mathbb{M}} L_2$ has the obvious meaning that $L_1 \leq_{\mathbb{M}} L_2$ while not $L_2 \leq_{\mathbb{M}} L_1$.

For a precise definition of the notion of *succinctness* we refer the reader to [12] and [13]. Here we formulate a sufficient condition for one logic to be exponentially more succinct than another which is inspired by [9, Lemma 1].

Definition 3. *Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M}\rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M}\rangle$ be two logics and let f be a strictly increasing polynomial. We say that L_1 is exponentially more succinct than L_2 on $\mathbb{M}$, and write $L_1 \preceq^{EXP}_{\mathbb{M}} L_2$, if for every $n \in \mathbb{N}$, there are two formulae $\alpha_n \in \Phi_1$ and $\beta_n \in \Phi_2$ satisfying the properties:*

1. *$|\alpha_n| \leq f(n)$ while $|\beta_n| \geq 2^{f(n)}$;*
2. *β_n is the shortest formula in Φ_2 with $\alpha_n \equiv_{\mathbb{M}} \beta_n$.*

In this case, we will also say that $\Delta = \{\alpha_n \mid n \in \mathbb{N}\}$ is more succinct than L_2 on $\mathbb{M}$ and write $\Delta \preceq_{\mathbb{M}} L_2$.

Therefore, if we want to show that L_1 is more succinct than L_2 on $\mathbb{M}$, it is sufficient to show that there is a sequence of formulae α_n in Φ_1 whose length grows polynomially, while the length of formulae β_n in Φ_2 that are $\mathbb{M}$-equivalent to α_n grows at least exponentially. Our definition is more restrictive than [9, Lemma 1]: in the latter, f does not need to be polynomial, and moreover, the functions that bind α_i and which β_i 'exponentially exceeds' need not be the same. On the other hand, in [12, 13] and [9], the condition $L_1 \leq_{\mathbb{M}} L_2$ is a prerequisite for $L_1 \preceq^{EXP}_{\mathbb{M}} L_2$ to be defined. We loosen that: we allow cases where a logic L_1 is exponentially more succinct than L_2 on a class of models $\mathbb{M}$ even if $L_1 \not\leq_{\mathbb{M}} L_2$: we require that the witnesses $\alpha_i \in L_1$ have an $\mathbb{M}$-equivalent in L_2, but this does not necessarily have to be the case for every $\varphi \in L_1$.

The difficult part of proving a succinctness result based on Definition 3 is often to show that β_n is indeed the *shortest* formula in Φ_2 that is equivalent to α_n on $\mathbb{M}$. This point highlights the importance of the models in $\mathbb{M}$: It may well be that for $\mathbb{M} \subseteq \mathbb{N}$, we have $L_1 \preceq^{EXP}_{\mathbb{N}} L_2$ but not $L_1 \preceq^{EXP}_{\mathbb{M}} L_2$: There may be just 'more candidates' for formulae to be equivalent to β_n in $\mathbb{M}$ than there are in $\mathbb{N}$. For example, Lutz [15] showed in 2006 that $\Phi_{[\varphi]ML} \preceq^{EXP}_{\mathbb{K}} \Phi_{ML}$, and this was only strengthened to $\Phi_{[\varphi]ML} \preceq^{EXP}_{\mathbb{S}5} \Phi_{ML}$ by French et al. in 2013 [9].

Formula size games.

In order to deal with the difficulties stemming from the second item of Definition 3, for every logic $[X]ML$ with $[X] \in \{[i], [\cap], [\cup], [\exists], [\cap, \cup]\}$, we define a suitable variant $FSG_{[X]ML}$ of Formula Size Games, first introduced in the seminal [1] as a generalisation of Ehrenfeucht-Fraïssé games for first-order logic. Here, we follow [9] and formulate these games in their one-player versions (their FSG corresponds to our $FSG_{[i]ML}$). Intuitively, the only player, called Spoiler, is presented with two sets of pointed Kripke models $\mathbb{A}$ and $\mathbb{B}$. Spoiler can win the $FSG_{[X]ML}$ on $\mathbb{A}$ and $\mathbb{B}$ if and only if there is a property Q_φ that is expressible with a formula $\varphi \in \Phi_{[X]}$, i.e., φ is true in all models in $\mathbb{A}$, and false in all models of $\mathbb{B}$. Moreover, the size of φ corresponds with the number of moves needed for Spoiler to win $FSG_{[X]ML}$.

Definition 4 (Formula Size Games). *The one- person (called Spoiler) formula size game $FSG_{[X]ML}$ on two sets of pointed models $\mathbb{A}$ and $\mathbb{B}$ is played as follows. During the course of the game, a game tree is constructed in such a way that each node is labelled with a pair $\langle \mathbb{C}, \mathbb{D}\rangle$ of sets of pointed models and one symbol from the set $\Sigma = \{p, \neg, \vee, [i], [\cup_\Gamma], [\cap_\Gamma], [\exists_\Gamma]\}$. A node labelled with the pair $\langle \mathbb{C}, \mathbb{D}\rangle$ is denoted $\langle \mathbb{C} \circ \mathbb{D}\rangle$. The models in $\mathbb{C}$ are called the models on the left, and $\mathbb{D}$ are called the models on the right.*

A node in the tree can be declared either open *or* closed. *Once a node has been declared "closed", no further game-moves can be played at it. The game begins with the root of the game tree $\langle \mathbb{A} \circ \mathbb{B}\rangle$ that is declared "open".*

Let an open node $\langle \mathbb{C} \circ \mathbb{D}\rangle$ be given. Spoiler can make one of the following moves at this node:

atomic-move: *Spoiler chooses a propositional symbol p such that $\mathbb{C} \models p$ and $\mathbb{D} \models \neg p$. The node is declared closed and labelled with the symbol p.*

not-move: *Spoiler labels the node with the symbol $\neg$ and adds one new open node $\langle \mathbb{D} \circ \mathbb{C}\rangle$ as a successor to the node $\langle \mathbb{C} \circ \mathbb{D}\rangle$.*

or-move: *Spoiler labels the node with the symbol $\vee$ and chooses two subsets $\mathbb{C}_1 \subseteq \mathbb{C}$ and $\mathbb{C}_2 \subseteq \mathbb{C}$ such that $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$. Two new open nodes are added to the tree as successors to the node $\langle \mathbb{C} \circ \mathbb{D}\rangle$, namely $\langle \mathbb{C}_1 \circ \mathbb{D}\rangle$ and $\langle \mathbb{C}_2 \circ \mathbb{D}\rangle$.*

$[i]$-move: *Spoiler labels the node with the symbol $[i]$ and, for each pointed model $(\mathscr{D}, t) \in \mathbb{D}$, he chooses a pointed model $(\mathscr{D}, t')$ such that tR_it' (if for some $(\mathscr{D}, v) \in \mathbb{D}$ this is not possible, Spoiler cannot play this move). All these new pointed models are collected in the set $\mathbb{D}_1$. A set of models $\mathbb{C}_1$ is then constructed as follows. For each pointed model $(\mathscr{C}, s) \in \mathbb{C}$, all the possible pointed models $(\mathscr{C}, s')$ such that sR_is' are added to $\mathbb{C}_1$. If for some $(\mathscr{C}, s)$, the point s does not have an R_i-successor, nothing is added to $\mathbb{C}_1$ for the pointed model $(\mathscr{C}, s)$. A new open node $\langle \mathbb{C}_1 \circ \mathbb{D}_1\rangle$ is added as a successor to the node $\langle \mathbb{C} \circ \mathbb{D}\rangle$. In this case, we also say that Spoiler has played an index-move.*

$[\cup_\Gamma]$-move: *Spoiler labels the node with the symbol $[\cup_\Gamma]$ and for each pointed model $(\mathscr{D}, t) \in \mathbb{D}$, he chooses a pointed model $(\mathscr{D}, t')$ such that tR_it' for some $i \in \Gamma$ (if there is a $(\mathscr{D}, t) \in \mathbb{D}$ for which this is not possible, then Spoiler cannot play this move). All these new pointed models are collected in the set $\mathbb{D}_1$. A set of models $\mathbb{C}_1$ is then constructed as follows. For each pointed model $(\mathscr{C}, s) \in \mathbb{C}$, and for each $i \in \Gamma$, all the possible pointed models $(\mathscr{C}, s')$ such that sR_is' are added to $\mathbb{C}_1$. A new open node $\langle \mathbb{C}_1 \circ \mathbb{D}_1\rangle$ is added as a successor to the node $\langle \mathbb{C} \circ \mathbb{D}\rangle$. In this case, we also say that Spoiler has played a $[\cup_\Gamma]$-move.*

$[\cap_\Gamma]$-move: *Spoiler labels the node with the symbol $[\cap_\Gamma]$ and, for each pointed model $(\mathscr{D}, t) \in \mathbb{D}$, he chooses a pointed model $(\mathscr{D}, t')$ such that tR_it' for all $i \in \Gamma$ (if for some $(\mathscr{D}, t) \in \mathbb{D}$ this is not possible, Spoiler cannot play this move). All these new pointed models are collected in the set $\mathbb{D}_1$. A set of models $\mathbb{C}_1$ is then constructed as follows. For each pointed model $(\mathscr{C}, s) \in \mathbb{C}$, all the possible pointed models $(\mathscr{C}, s')$ such that sR_is' for all $i \in \Gamma$ are added to $\mathbb{C}_1$.*

If for some $(\mathscr{C}, s)$, the point w does not have a successor for all $i \in \Gamma$, nothing is added to $\mathbb{C}_1$ for the pointed model $(\mathscr{C}, s)$. A new open node $\langle \mathbb{C}_1 \circ \mathbb{D}_1 \rangle$ is added as a successor to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$. In this case, we also say that Spoiler has played an $[\cap_\Gamma]$-move.

$[\exists_\Gamma]$**-move:** *Spoiler labels the node with the symbol $[\exists_\Gamma]$ and, for each pointed model $(\mathscr{D}, t) \in \mathbb{D}$ and each $i \in \Gamma$, he chooses a pointed model $(\mathscr{D}, t')$ such that tR_it' (if for some $(\mathscr{D}, t) \in \mathbb{D}$ this is not possible, Spoiler cannot play this move). All these new pointed models are collected in the set $\mathbb{D}_1$. A set of models $\mathbb{C}_1$ is then constructed as follows. For each pointed model $(\mathscr{C}, s) \in \mathbb{C}$, Spoiler chooses an $i \in \Gamma$ and all the possible pointed models $(\mathscr{C}, s')$ where sR_is' are added to $\mathbb{C}_1$. If for some $(\mathscr{C}, s)$, Spoiler has chosen an $i \in \Gamma$ such that the point s does not have an i-successor then nothing is added to $\mathbb{C}_1$ for the pointed model $(\mathscr{C}, s)$. A new open node $\langle \mathbb{C}_1 \circ \mathbb{D}_1 \rangle$ is added as a successor to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$. In this case, we also say that Spoiler has played an $[\exists_\Gamma]$-move.*

The game in which only atomic-, not-, or-, and $[i]$-moves are allowed is denoted $FSG_{[i]ML}$, extending this with $[\cup_\Gamma]$-moves we obtain $FSG_{[\cup]ML}$ and analogously, we use the notation $FSG_{[\cap]ML}$, $FSG_{[\exists]ML}$ and also $FSG_{[\cap,\cup]ML}$, all with obvious meaning.

Definition 5 (Winning Condition of $FSG_{[X]ML}$). *Spoiler wins the $FSG_{[X]ML}$ starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves if and only if there is a game tree T with root $\langle \mathbb{A} \circ \mathbb{B} \rangle$ and precisely n nodes such that every leaf of T is closed.*

Theorem 6. *Spoiler can win the $FSG_{[X]ML}$ starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves if and only if there is a formula $\varphi \in \Phi_X$ such that $\mathbb{A} \models \varphi$, $\mathbb{B} \models \neg\varphi$, and $|\varphi| = n$.*

For a proof of this theorem in the case of $FSG_{[i]ML}$, we refer the interested reader to [9]. Extending the proof to the cases $[\cap]$ML, $[\cup]$ML, $[\exists]ML$ and $[\cap, \cup]ML$ is straightforward.

Theorem 6 will be used as follows. In order to prove that $L_1 \preceq^{EXP}_{\mathbb{M}} L_2$, we will, for every n, construct two sets of pointed models $\mathbb{A}^n \subseteq \mathbb{M}$ and $\mathbb{B}^n \subseteq \mathbb{M}$ such that $\mathbb{A}^n \models \alpha_n$ and $\mathbb{B}^n \models \neg\alpha_n$ for some formula α_n from L_1; moreover, we show that

- the length of the formulae in the sequence $\alpha_1, \alpha_2, \ldots$ is bounded from above by a linear function in their indices;
- the least number of moves that Spoiler needs to win the FSG_{L_2} starting at $\langle \mathbb{A}^n \circ \mathbb{B}^n \rangle$ is 2^n.

Of course, we are facing the problem of proving a lower bound on the number of steps that Spoiler needs in order to win FSG_{L_2}. Here is one way of doing this that is best explained via an example for $FSG_{[i]ML}$. Suppose that Spoiler has to play an $FSG_{[i]ML}$ that starts with a node $\eta = \langle (\{\mathscr{C}, c), (\mathscr{D}, d)\} \circ \{(\mathscr{E}, e)\} \rangle$. If, in order to win, he must play a $[i]$-move, but, when played at η such a move inevitably leads to a node $\langle (\{\mathscr{C}_1, c_1), (\mathscr{D}_1, d_1)\} \circ \{(\mathscr{E}_1, e_1)\} \rangle$, where *one* of the pointed models on the left satisfy the same $\Phi_{[i]}$ formulae as the model on the right $(\mathscr{E}_1, e_1)$, then obviously, Spoiler cannot win the game from this node, because there is no formula α that is true in both $(\mathscr{C}_1, c_1), (\mathscr{D}_1, d_1)$ and false in $(\mathscr{E}_1, e_1)$. (A sufficient condition for Kripke models to satisfy the same formulas is that they are *bisimular*, see for instance [5]).

In order to avoid such a loosing position, Spoiler needs to first 'split' $(\mathscr{C}, c)$ and $(\mathscr{D}, d)$ (i.e., he needs to play an or-move) which will lead to adding two new branches to the game tree, namely one branch starting from the node η and going through the node $\langle \{(\mathscr{C}, c)\} \circ \{(\mathscr{E}, e)\} \rangle$ and another branch starting again from η and going through the node $\{(\mathscr{D}, d)\} \circ \{(\mathscr{E}, e)\}\rangle$. This means that we have shown a lower bound of 2 on any closed game tree with root $\langle \mathbb{C} \cup \{(\mathscr{C}, c), (\mathscr{D}, d)\} \circ \{(\mathscr{E}, e)\} \cup \mathbb{E} \rangle$. Of course, going back to our example, having played the *or*-move Spoiler can choose the one node from the nodes $\langle (\{\mathscr{C}, c)\} \circ \{(\mathscr{E}, e)\} \rangle$ and $\langle \{(\mathscr{D}, d)\} \circ \{(\mathscr{E}, e)\} \rangle$ whose successor is not a losing position when the necessary $[i] - move$ is played at it. This intuition was formalised in [9] as follows. Let $\mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$ be the set of all closed game trees for $FSG_{[X]ML}$ with root $\langle \mathbb{A} \circ \mathbb{B} \rangle$. A branch B in a closed game tree $T \in \mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$ is a path leading from the root of the tree to a closed leaf. Let $Br(T)$ be the set of branches of a closed game tree T. Two branches $B_1 = \eta_0, \eta_1 \ldots \eta_k$ and $B_2 = \eta'_0, \eta'_1, \ldots, \eta'_l$ in T are called *isomorphic* if $k = l$ and for every $i \leq k$, the labelling of η_i and η'_i with a symbol from $\{p, \neg, \vee, [i], [\cap_\Gamma], [\cup_\Gamma]\}$ is the same. If B_1 and B_2 are isomorphic, we write $B_1 \cong B_2$.

The next theorem was proven in [9] (it is true for game trees, and does not depend on the specific $FSG_{[X]ML}$).

Theorem 7 (Principle of Diverging Pairs). *[9, Theorem 2] Let $T \in \mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$. If $\mathbb{A}_1$, $\mathbb{A}_2$, ..., $\mathbb{A}_k$ are subsets of $\mathbb{A}$ and $\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_k$ are subsets of $\mathbb{B}$ and for every k trees $T_1 \in \mathcal{T}(\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle)$, $T_2 \in \mathcal{T}(\langle \mathbb{A}_2 \circ \mathbb{B}_2 \rangle)$, ..., $T_k \in \mathcal{T}(\langle \mathbb{A}_k \circ \mathbb{B}_k \rangle)$, there are k branches $B_1 \in Br(T_1)$, $B_2 \in Br(T_2)$, ..., $B_k \in Br(T_k)$ such that $B_i \not\cong B_j$ for all $1 \leq i < j \leq k$, then T contains at least k different branches.*

3. RESULTS

Overview of Some Known Succinctness Results.

We begin by saying a few words about the expressivity of the logics defined above. Since $[\cup_\Gamma]\varphi$ is equivalent to $\bigwedge_{i\in\Gamma}[i]\varphi$, and, likewise, $[\exists_\Gamma]\varphi$ is equivalent to $\bigvee_{i\in\Gamma}[i]\varphi$, we have that $[i]ML =_{\mathbb{K}} [\cup]ML =_{\mathbb{K}} [\exists]ML$: those logics are equally expressive on $\mathbb{K}$, and hence, on any $\mathbb{M} \subseteq \mathbb{K}$.

Although the logic $[\varphi]ML$ has a different 'feel' than that of modal logic that stems from its ability to make restrictions to sub-models, Plaza proved in [16] that occurrences of the $[\varphi]$-operator can be 'pushed-inside' the formula following it, and once reaching an atom, it can be eliminated. This is shown in the equivalences below:

$$
\begin{array}{lll}
(\mathscr{M}, w) \models [\varphi]p & \text{iff} & (\mathscr{M}, w) \models \varphi \to p; \\
(\mathscr{M}, w) \models [\varphi](\psi_1 \vee \psi_2) & \text{iff} & (\mathscr{M}, w) \models [\varphi]\psi_1 \vee [\varphi]\psi_2; \\
(\mathscr{M}, w) \models [\varphi]\neg\psi & \text{iff} & (\mathscr{M}, w) \models \varphi \to \neg[\varphi]\psi; \\
(\mathscr{M}, w) \models [\varphi][i]\psi & \text{iff} & (\mathscr{M}, w) \models \varphi \to [i][\varphi]\psi; \\
(\mathscr{M}, w) \models [\varphi_1][\varphi_2]\psi & \text{iff} & (\mathscr{M}, w) \models [\varphi_1 \wedge [\varphi_1]\varphi_2]\psi.
\end{array}
$$

Therefore, we have $[\varphi]ML =_{\mathbb{M}} [i]ML =_{\mathbb{M}} [\cup]ML =_{\mathbb{M}} [\exists]ML$ for any $\mathbb{M} \subseteq \mathbb{K}$. However, we are in a different situation with respect to $[\cap]ML$, i.e., $[\cap]ML$ is strictly more expressive than $[i]ML$ ([5]) and hence than any other of the above mentioned logics.

We are now in a position to summarise some of the already known succinctness results that are relevant to our exposition. The reader finds proofs for the following in [17] and [9].

THEOREM 8.

1. ([17]) *Suppose that* $|P| \geq 1$ *and* $|I| \geq 2$. *Then*
$$[\exists]ML \preceq^{EXP}_{\mathbb{K}} [\cup]ML \ \& \ [\cup]ML \preceq^{EXP}_{\mathbb{K}} [\exists]ML.$$

2. ([9]) *If* $|P| \geq 3$ *and* $|I| \geq 4$, *then*
$$[\exists]ML \preceq^{EXP}_{\mathbb{S}5} ML; \ [\cup]ML \preceq^{EXP}_{\mathbb{S}5} ML; \ [\varphi]ML \preceq^{EXP}_{\mathbb{S}5} ML.$$

The first item shows that it is possible for two logics to be more succinct than each other. This may seem surprising at first sight but it simply means that some properties are more succinctly expressed by using a modality for existential quantification, while others are expressed more succinctly by using the union of the accessibility relations. The results of the second item are strong in a sense that they are proven with respect to the class of models $\mathbb{S}_5$. This fact makes them significant for epistemic logic where $\mathbb{S}5$ is the dominant semantics. As an aside, note that there are some mild restrictions on the number of atoms and indices needed: as shown in [9] for item 2 of Theorem 8, the restrictions can be weakened for the semantics $\mathbb{K}$, and it is not known whether they are tight in all cases. It is worth pointing out that the results of item 2 remain true for weaker semantics, e.g., $\mathbb{K}$ or the class of models $\mathbb{KD}45$ which is used as a standard interpretation for belief.

We now list our main technical contribution.

THEOREM 9. *The following items are true:*

1. $[\varphi]ML \preceq^{EXP}_{\mathbb{K}} [\cap]ML$;
2. $[\varphi]ML \preceq^{EXP}_{\mathbb{K}} [\cup]ML$;
3. $[\varphi]ML \preceq^{EXP}_{\mathbb{K}} [\exists]ML$;
4. $[\cap]ML \preceq^{EXP}_{\mathbb{K}} [\cup]ML$;
5. $[\cup]ML \preceq^{EXP}_{\mathbb{K}} [\cap]ML$;
6. $[\varphi]ML \preceq^{EXP}_{\mathbb{K}} [\cap,\exists]ML$;
7. $[\cup]ML \preceq^{EXP}_{\mathbb{K}} [\cap,\exists]ML$;

We present the proof of this theorem in the next section. Because of the fact that $[\cap]ML$ and $[\cap,\exists[ML$ are equally expressive on Kripke models, we have some dependencies in the above theorem: in particular, item 6 of Theorem 9 implies item 1 and item 7 implies item 5, which is easily checked using Definition 3.

4. PROOF OF THEOREM 9

The formulae and the models.

We begin our proof of Theorem 9 by first defining the sequence of formulae and the sets of models we need in order to apply Definition 3. Since our results presuppose signatures that contain at least two relational indices and at least one propositional symbol, we fix one such signature $S = (I, P)$, where $I = \{a, d\}$ and $P = \{\mathbf{b}\}$, and all formulae we consider are formulae in the signature S. The reader can find the following mnemonic useful:

a stands for "solid arrow";
d stands for "dashed arrow";
$\mathbf{b}$ stands for "black node".

Let the sets of formulae $\Gamma \subset \Phi_{[\cup]ML}$ and $\Delta \subset \Phi_{[\varphi]ML}$ consist of the formulae γ_i and δ_i defined in Table 1.

We make the following observations about Γ and Δ. Intuitively, each formula $\gamma_i \in \Gamma$ says that there is a point v satisfying $\mathbf{b}$ that is n steps away from the current point and lies on a path consisting of only a and d steps. It is easy to see that for each γ_i, there is an equivalent ML-formula α_i defined recursively as follows $\alpha_1 = \langle a\rangle\mathbf{b} \vee \langle d\rangle\mathbf{b}$ and $\alpha_{i+1} = \langle a\rangle\alpha_i \vee \langle b\rangle\alpha_i$. Clearly the length of each α_i is at least 2^i. We are going to show that there are no shorter ML-formulae that are equivalent to the formulae from Γ. Similarly, using the translation from the previous section, we see that for every $\delta_i \in \Delta$, there is an equivalent ML-formula β_i defined recursively as follows $\beta_1 = \delta_1$ and $\beta_{i+1} = \beta_i \wedge (\langle a\rangle(\mathbf{b} \wedge \beta_i) \vee \langle d\rangle(\mathbf{b} \wedge \beta_i))$. Again we are going to show that none of the formulae in Δ has an equivalent in Φ_{ML} that is shorter than the respective formula β_i. To this end, we define a suitable set of models below.

Γ	Δ
$\gamma_1 \ \neg[\cup_{\{a,d\}}]\neg\mathbf{b}$	$\delta_1 \ \langle a\rangle\mathbf{b} \vee \langle d\rangle\mathbf{b}$
$\gamma_2 \ \neg[\cup_{\{a,d\}}][\cup_{\{a,d\}}]\neg\mathbf{b}$	$\delta_2 \ \langle\delta_1\rangle(\langle a\rangle\mathbf{b} \vee \langle d\rangle\mathbf{b})$
$\vdots$	$\vdots$
$\gamma_n \ \neg\underbrace{[\cup_{\{a,d\}}]\ldots[\cup_{\{a,d\}}]}_{n \text{ times}}\neg\mathbf{b}$	$\delta_n \ \langle\delta_{n-1}\rangle(\langle a\rangle\mathbf{b} \vee \langle d\rangle\mathbf{b})$
$\vdots$	$\vdots$

Table 1: The sets of formulae Γ and Δ.

DEFINITION 10 (THE MODELS $\mathbb{A}^n$ AND $\mathbb{B}^n$). *For every natural number $n \geq 1$, the sets of pointed models $\mathbb{A}^n$ (containing 2^n different models) and $\mathbb{B}^n$ (containing a single model) are defined as follows.*

($n = 1$) *The set $\mathbb{A}^1$ consists of the two pointed models $(\mathscr{A}^1_a, \alpha^1_a)$ and $(\mathscr{A}^1_d, \alpha^1_d)$ shown on the left of the dotted line in Figure 1. The set $\mathbb{B}^1$ contains only one pointed model namely, $(\mathscr{B}^1, \beta^1)$ that is shown on the right of the dotted line.*

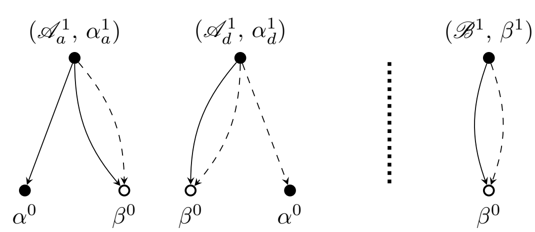

Figure 1: The sets of pointed models $\mathbb{A}^1$ and $\mathbb{B}^1$.

The black nodes satisfy the proposition $\mathbf{b}$ *whereas the white nodes do not. The subscripts in the names of the pointed Kripke models encode the way a black node can be reached from the uppermost node which is denoted by α with the relevant subscripts and superscripts in the case of the models in $\mathbb{A}^1$ and by β with a superscript in the case of the model in $\mathbb{B}^1$. For example, in the model $(\mathscr{A}^1_a, \alpha^1_a)$, a black node (namely, α^0) can be reached from α^1_a by making one step along the relation R_a represented by the arrow connecting these two nodes. In the model $(\mathscr{A}^1_d, \alpha^1_d)$, a black node can be reached from the node α^1_d by making one step along the relation R_d represented by the dashed arrow connecting the two nodes.*

($n+1$) *The set $\mathbb{A}^{n+1}$ consists of all the models built from the models in $\mathbb{A}^n \cup \mathbb{B}^n$ as shown in the Figure 2 below on* *the left of the dotted vertical line.*

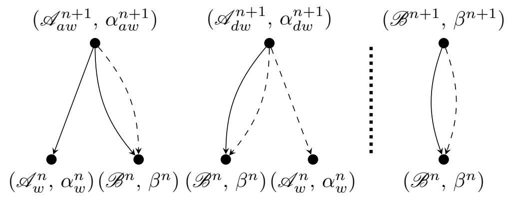

Figure 2: The sets of models $\mathbb{A}^{n+1}$ and $\mathbb{B}^{n+1}$.

For any pointed model $(\mathscr{A}^n_w, \alpha^n_w) \in \mathbb{A}$, the pointed model $(\mathscr{A}^{n+1}_{aw}, \alpha^{n+1}_{aw})$ is obtained by taking a black node, denoted by α^{n+1}_{aw}, and connecting it to the point α^n_w in $\mathscr{A}^n_w$ and the point β^n in the model $\mathscr{B}^n$ as shown. The pointed model $(\mathscr{A}^{n+1}_{dw}, \alpha^{n+1}_{dw})$ is constructed in a similar fashion. The set $\mathbb{B}^{n+1}$ contains only the model $(\mathscr{B}^{n+1}, \beta^{n+1})$ shown on the right of the dotted line.

Intuitively, the subscript w and the superscript n in the pointed model $(\mathscr{A}^n_w, \alpha^n_w) \in \mathbb{A}^n$ say that there is a sequence of relation steps of length n encoded by w (a word over the alphabet $\{a, d\}$), leading from the uppermost point of $\mathscr{A}^n_w$, i.e., α^n_w, to the only lowermost node satisfying the proposition **b**. Since there are 2^n different words w of length n over the alphabet $\{a, d\}$ and for every such w, there is a corresponding pointed model in $\mathbb{A}^n$, this means that $\mathbb{A}^n$ contains 2^n different pointed models.

Example 11. *The models $(\mathscr{A}^2_{aa}, \alpha^2_{aa})$ and $(\mathscr{B}^2, \beta^2)$ are shown in Figure 3 below. Note how the pointed models $\mathscr{A}^1_a$ and $\mathscr{B}^1$ are used in the construction of $\mathscr{A}^2_{aa}$.*

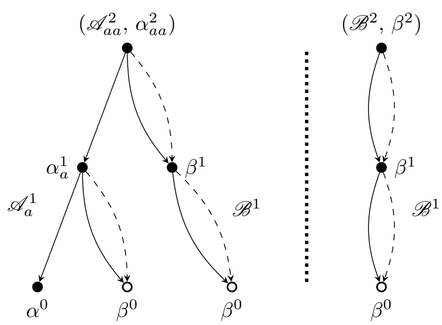

Figure 3: The models $(\mathscr{A}^2_{aa}, \alpha^2_{aa})$ and $(\mathscr{B}^2, \beta^2)$.

Intuitively, the second item from the next theorem says that $[\exists_{\{a,d\}}]$ or $[\cap_{\{a,d\}}]$- moves are powerless on these sets of pointed models.

Theorem 12. *The following are true.*

1. *$\mathbb{A}^n \models \gamma_n$ & $\mathbb{A}^n \models \delta_n$ whereas $\mathbb{B}^n \models \neg\gamma_n$ & $\mathbb{B}^n \models \neg\delta_n$.*
2. *Spoiler cannot play a $[\exists_{\{a,d\}}]$ or $[\cap_{\{a,d\}}]$-move at a node of the form $\langle(\mathscr{A}^n_w, \alpha^n_w) \circ (\mathscr{B}^n, \beta^n)\rangle$ or $\langle(\mathscr{B}^n, \beta^n) \circ (\mathscr{A}^n_w, \alpha^n_w)\rangle$ without losing the game.*

Proof. The first item can be checked by using the remarks at the beginning of this section. The proof of the second item is best understood when given about the pointed models for $n+1$ (see Figure 2). Let us suppose that we have a node with $(\mathscr{B}^{n+1}, \beta^{n+1})$ on one side and $(\mathscr{A}^{n+1}_w, \alpha^{n+1}_w)$ on the other. If Spoiler plays a $[\exists_{\{a,d\}}]$ or a $[\cap_{\{a,d\}}]$-move, he will need to select $(\mathscr{B}^n, \beta^n)$ as a successor to the pointed model $(\mathscr{A}^{n+1}_w, \alpha^{n+1}_w)$ and the same pointed model, namely, $(\mathscr{B}^n, \beta^n)$ as a successor of the pointed model $(\mathscr{B}^{n+1}, \beta^{n+1})$ thus reaching a node in the game tree with two bisimilar models, one on the left and one on the right. However, this is a losing position for Spoiler. $\square$

We are now set to demonstrate how we employ the principle of diverging pairs.

Theorem 13. *For every n, Spoiler needs at least 2^n moves to win the game $FSG_{[\cap,\exists]ML}$ (and hence the games $FSG_{[\cap]ML}$ and $FSG_{[\exists]ML}$) starting at a node $\langle \mathbb{A}^n \circ \mathbb{B}^n\rangle$.*

Proof. $\mathbb{A}^n$ contains 2^n different pointed models corresponding to the 2^n different words w over the alphabet $\{a, d\}$. Thus, we have 2^n different pairs $\langle(\mathscr{A}^n_w, \alpha^n_w), (\mathscr{B}^n, \beta^n)\rangle$. Let us consider an arbitrary $T \in \mathcal{T}(\langle\{(\mathscr{A}^n_w, \alpha^n_w)\} \circ \{(\mathscr{B}^n, \beta^n)\}\rangle)$. For a branch B in T let $[i_1][i_2]\ldots[i_k]$ be the possibly empty sequence of index moves that occur along B when we traverse the branch from the root $\langle\{(\mathscr{A}^n_w, \alpha^n_w)\} \circ \{(\mathscr{B}^n, \beta^n)\}\rangle$ to its leaf. Let $I(B)$ denote the sequence $i_1, i_2, \ldots i_k$, i.e., the sequence of indices from $[i_1][i_2]\ldots[i_k]$. Therefore, $I(B)$ is a word over the alphabet $\{a, d\}$. The proof of the theorem will follow from Theorem 7 if we show that T contains a branch B_w such that $I(B_w) = w$, because if $w_1 \neq w_2$ then two branches B_{w_1} and B_{w_2} such that $I(B_{w_1}) = w_1$ and $I(B_{w_1}) = w_2$ cannot be isomorphic.

We proceed by induction on n to prove the following statement. Any closed game tree for the game $FSG_{[\cap,\exists]ML}$ starting at a node $\langle(\mathscr{A}^n_w, \alpha^n_w) \circ (\mathscr{B}^n, \beta^n)\rangle$ or a node $\langle(\mathscr{B}^n_w, \beta^n) \circ (\mathscr{A}^n, \alpha^n_w)\rangle$, contains a branch B such that $I(B) = w$.

Let us first consider a game starting at $\langle(\mathscr{A}^1_a, \alpha^1_a)\circ(\mathscr{B}^1, \beta^1)\rangle$ (the case for $\langle(\mathscr{A}^1_d, \alpha^1_d) \circ (\mathscr{B}^1, \beta^1)\rangle$, or $\langle(\mathscr{B}^1_a, \beta^1) \circ (\mathscr{A}^1, \alpha^1_a)\rangle$ and, hence, for $\langle(\mathscr{B}^1_d, \beta^1) \circ (\mathscr{A}^1, \alpha^1_d)\rangle$ are analogous). It is obvious that Spoiler cannot begin the game with an atomic move so he can start by playing a number of not and or-moves This, however, will lead to at least one branch that consists of a number of nodes of the form $\langle(\mathscr{A}^n_w, \alpha^n_w)\circ(\mathscr{B}^n, \beta^n)\rangle$ or $\langle(\mathscr{B}^n, \beta^n)\circ(\mathscr{A}^n_w, \alpha^n_w)\rangle$. Since neither of these nodes can be closed, at some point, Spoiler will need to play an $[i]$, $[\cap_{\{a,d\}}]$ or $[\exists_{\{a,d\}}]$ move. We already know from Theorem 12 that Spoiler cannot play one of the latter two moves. He will not play a $[d]$-move either, because in the successor node in the branch, he would have two bisimilar models. So the only first move involving any index will be an $[a]$-move. Assume the statement is true for n and let us consider a game starting at $\langle(\mathscr{A}^{n+1}_w, \alpha^{n+1}_w) \circ (\mathscr{B}^{n+1}, \beta^{n+1})\rangle$ (the remaining cases are similar). Suppose that $w = aw'$ (the case $w = dw'$ is analogous). As in the case for $n = 1$, Spoiler can begin by playing a number of not- and or-moves, but, again as above, this will lead to at least one branch that consists of a number of nodes $\langle(\mathscr{A}^{n+1}_w, \alpha^{n+1}_w) \circ (\mathscr{B}^{n+1}, \beta^{n+1})\rangle$ or $\langle(\mathscr{B}^{n+1}, \beta^{n+1}) \circ (\mathscr{A}^{n+1}_w, \alpha^{n+1}_w)\rangle$. Therefore, at some point, he will need to play an $[i], [\cap_{\{a,d\}}]$ or $[\cup_{\{a,d\}}]$ move. But, as in the case for $n = 1$, the only possible such move is $[a]$. However, Spoiler cannot play an $[a]$-move at a node $\langle(\mathscr{A}^{n+1}_w, \alpha^{n+1}_w) \circ (\mathscr{B}^{n+1}, \beta^{n+1})\rangle$ because this will lead to the occurrence of two bisimilar models, one on the left and the other on the right in the successor node. Hence, Spoiler must play a $[a]$-move at a node of the form $\langle(\mathscr{B}^{n+1}, \beta^{n+1}) \circ (\mathscr{A}^{n+1}_w, \alpha^{n+1}_w)\rangle$. His only choice is to select the pointed model $(\mathscr{A}^n_{w'}, \alpha^n_{w'})$ on the right and $(\mathscr{B}^n, \beta^n)$ on the left.We apply the induction hypothesis which completes the proof. $\square$

Corollary 14. *Let $\mathbb{Z} = \cup_n(\mathbb{A}^n \cup \mathbb{B}^n)$. Then it follows from Theorem 12 and 13 that $\Delta \preceq^{EXP}_{\mathbb{Z}} [\cap]ML$, that $\Delta \preceq^{EXP}_{\mathbb{Z}} [\exists]ML$, and that $\Gamma \preceq^{EXP}_{\mathbb{Z}} [\cap]ML$. Moreover, we also have $\Gamma \preceq^{EXP}_{\mathbb{Z}} [\cap, \exists]ML$ and $\Delta \preceq^{EXP}_{\mathbb{Z}} [\cap, \exists]ML$. In turn, these claims prove items 1, 3, 5, 6 and 7 of Theorem 9, respectively.*

In order to prove the items 2 and 4 of Theorem 9 we continue by defining two new sets of formulae and a suitable set of pointed models as follows.

Let the set of formulae $\Psi \subset \Phi_{[\cap]ML}$ and $\Omega \subset \Phi_{[\varphi]ML}$ be defined recursively as shown in Table 2

Ψ	Ω
ψ_1 $[\cap_{\{a,d\}}]\mathbf{b}$	ω_1 $[a]\mathbf{b} \vee [d]\mathbf{b}$
ψ_2 $[\cap_{\{a,d\}}][\cap_{\{a,d\}}]\mathbf{b}$	ω_2 $\langle\omega_1\rangle(\langle a\rangle\mathbf{b} \wedge \langle d\rangle\mathbf{b})$
$\vdots$	$\vdots$
ψ_n $\underbrace{[\cap_{\{a,d\}}]\ldots[\cap_{\{a,d\}}]}_{n \text{ times}}\mathbf{b}$	ω_n $\langle\omega_{n-1}\rangle(\langle a\rangle\mathbf{b} \wedge \langle d\rangle\mathbf{b})$
$\vdots$	$\vdots$

Table 2: The sets of formulae Ψ and Ω.

The reader can easily check that for every i, the *ML*-formula ζ_i defined as $\zeta_1 = \omega_1$ and $\zeta_{i+1} = \zeta_i \wedge (\langle a\rangle(\mathbf{b} \wedge \zeta_i) \wedge \langle d\rangle(\mathbf{b} \wedge \zeta_i))$ is equivalent to the formula ω_i.

Following the conventions we used in Definition 10, for any $n \in \mathbb{N}$, we construct two sets of pointed models $\mathbb{O}^n$ and $\mathbb{P}^n$.

Definition 15 (The models $\mathbb{O}^n$ and $\mathbb{P}^n$). *The sets of pointed models $\mathbb{O}^n$ and $\mathbb{P}^n$, containing 2^n different models each, are built recursively as follows.*

($n = 1$) The set $\mathbb{O}^1$ consists of the two pointed models $(\mathscr{O}^1_a, o^1_a)$ and $(\mathscr{O}^1_d, o^1_d)$ shown on the left of the dotted line in Figure 4 below. The set $\mathbb{P}^1$ contains the models $(\mathscr{P}^1_a, \rho^1_a)$ and $(\mathscr{P}^1_d, \rho^1_d)$ shown on the right of the dotted line.

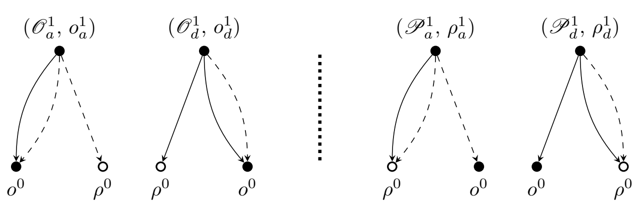

Figure 4: The models $\mathbb{O}^1$ and $\mathbb{P}^1$.

($n+1$) The sets $\mathbb{O}^{n+1}$ and $\mathbb{P}^{n+1}$ are built as shown in Figure 5. We follow an algorithmic pattern analogous to the one we used in the construction of the models in $\mathbb{A}^{n+1}$ and $\mathbb{B}^{n+1}$ from Definition 10. It should be clear that each of the sets $\mathbb{O}^n$ and $\mathbb{P}^n$ contains 2^n different pointed models - one for each subscript w of length n.

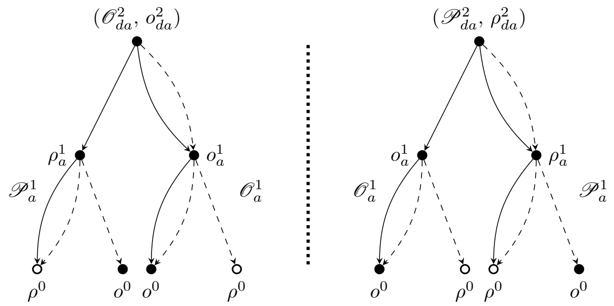

Figure 6: The models $(\mathscr{O}^2_{da}, o^2_{da})$ and $(\mathscr{P}^2_{da}, \rho^2_{da})$.

As before, for any pair of pointed models $(\mathscr{O}^n_w, o^n_w)$ and $(\mathscr{P}^n_w, \rho^n_w)$, we have that the subscript w encodes a sequence of n relation steps that lead from o^n_w to a black point o^0; the same sequence w leads from ρ^n_w to a white point ρ^0.

Example 16. *The pair of models $(\mathscr{O}^2_{da}, o^2_{da})$ and $(\mathscr{P}^2_{da}, \rho^2_{da})$ are shown in Figure 6 below. The subscript "da" and the superscript "2" in $(\mathscr{O}^2_{da}, o^2_{da})$ mean that starting at o^2_{da} and making one step along the relation R_d (represented by the dashed arrow) followed by a step along R_a (represented by the solid arrow), we arrive at a black point, i.e., a point satisfying the proposition $\mathbf{b}$. The same sequence of relation steps "da" leads to a white point, i.e., a point that does not satisfy the proposition $\mathbf{b}$ from ρ^2_{da} in the Kripke model $\mathscr{P}^2_{da}$. Note that $(\mathscr{O}^2_{da}, o^2_{da})$ and $(\mathscr{P}^2_{da}, \rho^2_{da})$ are bisimilar with respect to a, i.e., the relation R_a represented by the solid arrow.*

Analogously to Theorem 12, we have the next Theorem where the second item says that $[\cup_\Gamma]$ moves will not help Spoiler win a $FSG_{[\cup]ML}$.

Theorem 17. *The following are true.*

1. *$\mathbb{O}^n \models \psi_n \wedge \omega_n$ whereas $\mathbb{P}^n \models \neg\psi_n \wedge \neg\omega_n$.*
2. *Spoiler cannot play a $[\cup_{\{a,d\}}]$-move at a node of the form $(\mathscr{O}^n_w, o^n_w) \circ (\mathscr{P}^n_w, \rho^n_w)$ or $(\mathscr{P}^n_w, \rho^n_w) \circ (\mathscr{O}^n_w, o^n_w)$. without losing the game.*

Proof. The proof of both items is similar to the proofs of the analogous items from Theorem 12. $\square$

Before continuing, we would like to point out that, for any formula ω_i, there is an equivalent on $\mathbb{O}^n \cup \mathbb{P}^n$ formula $\eta_i \in \Phi_{ML}$ defined as follows. $\eta_1 = [a]\mathbf{b} \vee [d]\mathbf{b}$ and $\eta_i + 1 = [a]\eta_i \vee [d]\eta_i$. The proof of the next theorem mimics the proof of Theorem 13.

Theorem 18. *For every n, Spoiler needs at least 2^n moves to win the $FSG_{[\cup]ML}$, starting at a node $\langle\mathbb{O}^n \circ \mathbb{P}^n\rangle$.*

Corollary 19. *Let $\mathbb{Z} = \cup_n(\mathbb{O}^n \cup \mathbb{P}^n)$. Then it follows from Theorem 12 and 13 that $\Omega \preceq^{EXP}_{\mathbb{Z}} [\cup]ML$, and that $\Psi \preceq^{EXP}_{\mathbb{Z}} [\cup]ML$. In turn, these claims prove items 2, and 4 of Theorem 9, respectively.*

5. CONCLUSION

We have demonstrated succinctness results that compare a number of modal logics. Most of them were equally expressive as pure modal logic $[i]ML$, but we also considered

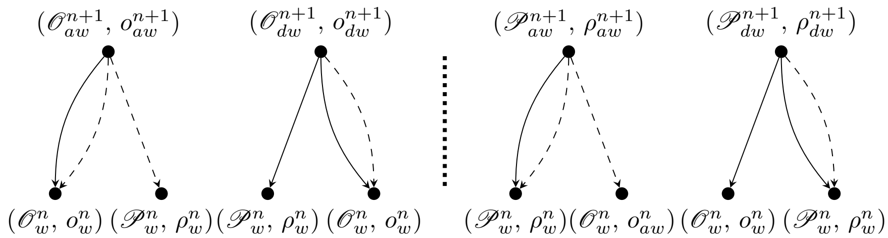

Figure 5: The sets of models $\mathbb{O}^n$ and $\mathbb{P}^n$ where $n > 1$.

a logic, $[\cap]ML$, that is more expressive than all the others. Interestingly, we showed that, although $[\cap]ML$ is more expressive than $[\cup]ML$, some properties are more succinctly expressed in the weaker logic, and others are more succinctly expressed in the stronger one.

Avenues for further research are plenty. All the relative succinctness results we presented are for $\mathbb{K}$. (The proofs show that they in fact hold for a smaller class: models with at most two successors for each i). It is an open question how easy they generalise to more specialised models, like $\mathbb{S}5$. In fact, all the models we presented in this paper are generated models, which made reasoning in the FSGs easier, because the result of playing a modal-move kept the models of interest on both sides of the node in a game tree 'synchronised'. However, if the accessibilities in the models receive more properties, things become more difficult: playing a $[i]$-move for instance forces us to take into account that the successor point in the next node may be the same (if R_a is reflexive) or goes back 'higher up in the model' (if R_a is symmetric).

Also, note that none of our results provides a system that is more succinct than $[\varphi]ML$. Although it is possible to define $FSG_{[\varphi]ML}$, such games are difficult to employ on concrete sets of models, because the attention not only goes to the current point and possible successors, but to arbitrary points in the model satisfying φ–moreover, Spoiler can choose the φ at will. Indeed, there are also many other modal logics where settling succinctness questions may need richer or indeed other tools than FSGs. An important example that comes to mind is where quantification over coalitions is introduced, like in epistemic logic [3] but also coalition logic [2].

Acknowledgements.

Petar Iliev acknowledges support from European Research Council grant EPS 313360.

6. REFERENCES

[1] M. Adler and N. Immerman. An $n!$ lower bound on formula size. *ACM ToCL*, 4(3):296–314, 2003.

[2] T. Ågotnes, W. van der Hoek and M. Wooldridge. Quantified Coalition Logic. *Synthese*, 165(2):233–258, 2008.

[3] T. Ågotnes, W. van der Hoek and M. Wooldridge. Quantifying over Coalitions in Epistemic Logic. AAMAS, pp. 665–672, 2008.

[4] F. Baader, D. Calvanese, D. McGuinness, D. Nardi, and P. Patel-Schneider (eds). *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, 2003.

[5] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[6] M. Cadoli, F. Donini, P. Liberatore, and M. Schaerf. The Size of a Revised Knowledge Base. *Artificial Intelligence*, 115:25–64, 1999.

[7] G. Gogic, H. Kautz, C. Papadimitriou, and B. Selman. The comparative linguistics of knowledge representation. IJCAI, pp. 862–869, 1995

[8] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge*. The MIT Press, 1995.

[9] T. French, W. van der Hoek, P. Iliev, and B. Kooi. On the succinctness of some modal logics. *Artificial Intelligence*, 197:56–85, 2013.

[10] G. Gargov, S. Passy, and T. Tinchev. Modal environment for boolean speculations. In D. Skordev, editor, *Proceedings of The Advanced International Summer School and Conference on Mathematical Logic and Its Applications*, pages 480–490, 1987.

[11] R. Goldblatt. *Logics of Time and Computation (CSLI Lecture Notes Number 7)*. CSLI, Stanford, CA, 1987.

[12] M. Grohe and N. Schweikardt. Comparing the succinctness of monadic query languages over finite trees. *Theoretical Informatics and Applications*, 38(4):343–373, 2004.

[13] M. Grohe and N. Schweikardt. The succinctness of first-order logic on linear orders. *LMCS*, 1:1–25, 2005.

[14] D. Harel. Dynamic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 497–604, 1984.

[15] C. Lutz. Complexity and succinctness of public announcement logic. *Proc. AAMAS*, pp. 137–144, 2006.

[16] J.A. Plaza. Logics of public communications. *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pp. 201–216, 1989.

[17] W. van der Hoek, P. Iliev, and B. Kooi. On the relative succinctness of two extensions by definitions of multimodal logic. CiE, LNCS 7318 , pp. 323–333, 2012.

[18] W. van der Hoek and M. Wooldridge. Logics for multi-agent systems. In G. Weiss, editor, *Multi-Agent Systems*, pp. 671–810. MIT Press, 2013.

[19] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*. Springer, 2007.