Complexity of Stability-based Solution Concepts in Multi-issue and MC-net Cooperative Games

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ABSTRACT

MC-nets constitute a natural compact representation scheme for cooperative games in multiagent systems. In this paper, we study the complexity of several natural computational problems that concern solution concepts such as the core, the least core and the nucleolus. We characterize the complexity of these problems for a variety of subclasses of MC-nets, also considering constraints on the game such as superadditivity (where appropriate). Many of our hardness results are derived from a hardness result that we establish for a class of multi-issue cooperative games (SILT games); we suspect that this hardness result can also be used to prove hardness for other representation schemes.

Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics

General Terms

Algorithms, Economics, Theory

Keywords

Cooperative games, core, computational complexity

1. INTRODUCTION

In settings with multiple self-interested agents, the agents can often benefit from forming *coalitions*, which allows them to accomplish tasks that they could not accomplish individually. *Cooperative game theory* provides tools to answer several important questions in this context, such as how the gains from such cooperation are to be distributed among the agents. This has led to significant and sustained interest from multiagent systems researchers in computational aspects of cooperative game theory. A book is now available on this topic [2].

The most commonly studied model in cooperative game theory specifies a *value* $v(S)$ for every subset $S \subseteq A$, where A is the set of agents. This is the value that the agents in S can obtain and distribute among themselves if they work (only) with each other. $v : 2^A \to \mathbb{R}$ is known as the *characteristic function* of the game. Several assumptions are inherent in this model: for example, more generally there may be restrictions on how agents can transfer utility among themselves, or the agents may care about the actions of agents outside the coalition. Nevertheless, it is a broadly applicable model and we will restrict attention to it in this paper.

The straightforward way to represent a cooperative game with n agents requires listing 2^n numbers, one for each coalition $S \subseteq A$. This is generally not feasible. Usually, however, there is structure in the game that allows us to represent it compactly. In this paper, we focus on representations whose compactness relies on the insight that the characteristic function is often a sum of multiple functions—that is, $v = \sum_t v_t$ for multiple *issues* t—where each v_t can be represented compactly. For example, if for each individual issue t, many (for example, all but a constant number of) are *dummy agents* (where i is a dummy agent for v_t if $v_t(S \cup \{i\}) = v_t(S)$ for all S), then we can explicitly specify the value of the function v_t for each subset of non-dummy agents [4]. Alternatively, *Marginal Contribution nets (MC-nets)* [14] specify, for each issue, a logical *pattern* such that some constant value is obtained if the coalition satisfies the pattern, and zero otherwise. Both of these representation schemes are fully expressive; necessarily, some games will require exponential space to specify, but many interesting families of games (e.g., graph games) can be specified compactly.

MC-nets in particular have received a significant amount of attention in recent years. Which games can be specified compactly using them depends on which logical operators are allowed in the patterns. The original paper [14] focused on the case where only conjunctions and negations (and no brackets) are allowed. More recently, it was extended to also allow disjunctions and brackets [8]. Inspired by these papers, our main objective in this paper is to systematically characterize how the complexity of solving cooperative games represented as MC-nets depends on what constraints are imposed on patterns – specifically, which logical operators are allowed and whether negative values are allowed. It turns out that a key step is to first consider general multi-issue games in which all issues but one concern only a small number of agents.

But what does it mean to *solve* a game? There are many solution concepts in cooperative game theory, such as the Shapley value [19], the kernel [6], the core [11], the least core [17] and the nucleolus [18]. Among these, the Shapley value and the core are particularly prominent. A key and defining property of the Shapley value is its additivity: an agent's Shapley value is the sum of its Shapley values in the individual issues, so that having multiple issues (or patterns) does not get in the way of computational tractability [4, 14, 8]. Therefore, in this paper, we focus on stability-based solution concepts, including the core, the least core and the nucleolus.

2. RELATED RESEARCH

Deng and Papadimitriou [7] proved that the CORE-EMPTINESS problem (see Definition 2) is NP-complete in graph games where agents are vertices and a coalition's value is the sum of all edges'

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weights in the subgraph induced by that coalition. As has been previously pointed out [14, 12], MC-nets can encode graph games efficiently, so the problem is also hard for MC-nets. However, a natural and extremely common constraint on the characteristic function is for it to be *superadditive*, that is, for $S \cap S' = \emptyset$, we must have $v(S \cup S') \geq v(S) + v(S')$. The intuition is that one course of action for any coalition is simply to further divide into two subcoalitions and take the sum of their values, so the (maximum) value the coalition can achieve must be at least this sum. Graph games are superadditive if and only if there are no negative edges, and if there are no negative edges the graph game must be convex and its core nonempty. So, if a superadditivity constraint is added, the CORE-EMPTINESS problem is trivial for graph games, and its complexity is not clear for MC-nets.

Conitzer and Sandholm [5] introduced a compact representation scheme that fundamentally relied on the game being superadditive. They proved that the CORE-EMPTINESS problem is coNP-complete under their representation. However, they proved that this hardness was strictly due to the hardness of computing the value of the grand coalition ($v(A)$) under their representation, because once that value is given, the CORE-EMPTINESS problem can be solved in polynomial time. Hence this result cannot imply hardness for MC-nets, where $v(A)$ is easy to compute. In other work [4], Conitzer and Sandholm consider the multi-issue representation described above (a constant number of non-dummy agents per issue) and proved that the harder NOT-IN-CORE problem (see Definition 3) is NP-complete even with a superadditivity constraint. However, they did not settle the complexity of the CORE-EMPTINESS problem under these conditions. In this paper, we prove that the CORE-EMPTINESS problem is in fact NP-complete under a multi-issue representation (with a constant number of non-dummy agents for all but one issue), even when requiring superadditivity. We show that this also implies hardness for several, but not all, variants of MC-nets. The proof is quite involved; we will give some intuition for why this might be necessary at the end of Section 3.2.

Note that our work is not the first to prove NP-hardness for the CORE-EMPTINESS problem in cooperative games that are superadditive. Greco et al. [13] proved such hardness for games that are specified using "polynomial-time worth functions". But their result does not seem to apply to MC-nets and multi-issue games, as their representation scheme is significantly more powerful. Specifically, in their hardness proof, the coalition value switches to some value once the coalition size exceeds $|A|/2$, and we do not see how this can be expressed using MC-nets with only limited logical operators or using multi-issue games with small issues. On the other hand, our result implies their hardness result, as SILT games are polynomial-time worth function games. To the best of our knowledge, no hardness results are known for CORE-EMPTINESS under MC-nets or multi-issue representations. Previous results left open the possibility that these problems were polynomial-time solvable under such representation schemes.

Although the core is a computationally challenging solution concept in many cases, some positive results are known. For example, the NOT-IN-CORE problem is in P for MC-nets if the treewidth of the corresponding agent graph is bounded [14]. This problem is also in P for graph (or hyper-graph) games if all the edges (or hyperedges) have non-negative values [7]. Those games correspond to MC-nets with only $\wedge$ operators and non-negative pattern values. In fact, Deng and Papadimitriou [7] gave an efficient network flow algorithm that can solve a harder problem: MOST-VIOLATED-COALITION (see Definition 4).[1] A similar network flow algorithm can be found in Lawler [15, p. 125]. (The provisioning problem there is identical to the MOST-VIOLATED-COALITION problem.) Those algorithms can further be used to compute an element of the least core, and, under certain conditions [9], the nucleolus, efficiently. The nucleolus is a very attractive solution concept; among other properties, it is unique and it lies within the core whenever the core is nonempty.

As becomes apparent from the above discussion, the three problems CORE-EMPTINESS, NOT-IN-CORE, and MOST-VIOLATED-COALITION are closely related to each other and to stability-based solution concepts like the core, the least core and the nucleolus. This makes it natural to study them all together, as we do in the rest of this paper.

3. PRELIMINARIES

In this section, we define the computational problems that we study, and review some basic results. We will not yet discuss how games are represented; the definitions of the computational problems are valid for any representation scheme (though of course their complexity depends on which scheme is used).

3.1 Problem Definitions

Given agents A, we first formally define the core:

DEFINITION 1 (CORE). *Let $x : A \to \mathbb{R}$ denote a payment vector and let $x(S) = \sum_{a \in S} x(a)$ be the total payment to coalition $S \subseteq A$. The* core *is the set of payment vectors with $x(A) = v(A)$ that pay every coalition at least its value, i.e., $\{x \mid x(A) = v(A) \wedge (\forall S \subseteq A) x(S) \geq v(S)\}$.*

We will study the following related decision problems:

DEFINITION 2. *In the* CORE-EMPTINESS *problem, we are given a cooperative game. The instance has a* yes *answer if and only if the core of that game is empty.*

DEFINITION 3. *In the* NOT-IN-CORE *problem, we are given a cooperative game and a payment vector x ($x(A) = v(A)$). The instance has a* yes *answer if and only if x is not in the core of that game (that is, there exists a* blocking coalition *$\emptyset \subsetneq S \subsetneq A$ such that $v(S) > x(S)$).*

DEFINITION 4. *In the* MOST-VIOLATED-COALITION *problem, we are given a cooperative game, a payment vector x ($x(A) = v(A)$), and a violation goal $\gamma \in \mathbb{R}$. The instance has a* yes *answer if and only if there exists $\emptyset \subsetneq S \subsetneq A$ such that $v(S) - x(S) > \gamma$.*

In Definitions 3 and 4, when the answer is *yes*, we might also like to *find* a coalition that proves that this is the case. Hence, the reader might prefer a definition of the computational problem that is more constructive. Fortunately, in both cases, we can use an algorithm for the decision problem to actually construct the coalition in question, shown as Algorithm 1.[2] In the case of Definition 4, we may also wish to find the maximum γ with answer *yes*; searching for this γ up to an arbitrarily good approximation is straightforward by binary search.

[1] They do not explicitly state that their algorithm can solve this problem, but it does. Specifically, the maximum violation is $v(A)$ (or equivalently $v(N) = x(N)$ in their paper's notation) minus the value of the maximum flow/minimum cut.

[2] The algorithm is correct for the following reasons. If the decision algorithm returns *yes* after $x(a)$ is increased by M, there must be a subset S with $a \notin S$ that can prove the *yes* answer; hence we can safely exclude a by not restoring $x(a)$. If the decision algorithm returns *no* after $x(a)$ is increased by M (but before that the answer is *yes*), all S that can prove the *yes* answer (without using what has already been excluded) must include a; therefore we restore $x(a)$ (so the answer remains to be *yes*) and put that a into C.

Algorithm 1 Given an algorithm that decides whether a coalition S with $v(S) - x(S) > 0$ (or $v(S) - x(S) > \gamma$) exists, and a game for which the answer is *yes*, construct a coalition with that property.

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for all a ∈ A do
    increase x(a) and v(N) by M = (max_S v(S)) − γ
    run the algorithm for the decision variant again;
    if the answer is no then
        restore x(a), v(N) to their original values
    end if
end for
Output the coalition C = {a ∈ A : x(a) < M}.
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3.2 Basic Results

What could be a certificate for the core being empty? The well-known Bondareva-Shapley theorem [1, 20] provides the answer.

THEOREM 1 (BONDAREVA-SHAPLEY THEOREM). *For any cooperative game with non-negative values ($v(S) \geq 0$ for all $S \subseteq A$), the core is empty if and only if there exists a weight function $w : 2^A \setminus \{\emptyset\} \to [0, 1]$ such that*

$$\sum_{\emptyset \subset S \subseteq A} w(S)v(S) > v(A) \tag{1}$$

$$(\forall a \in A) \sum_{\emptyset \subset S \subseteq A: a \in S} w(S) \leq 1 \tag{2}$$

The following lemma is also well-known (see, e.g., [16, 10]).

LEMMA 1. *If the value of any coalition $v(S)$ can be computed in polynomial time, then the* CORE-EMPTINESS*,* NOT-IN-CORE*, and* MOST-VIOLATED-COALITION *problems are in NP.*

An efficient algorithm for one of the problems sometimes leads to one for another. The NOT-IN-CORE problem is the special case of the MOST-VIOLATED-COALITION problem where $\gamma = 0$. Meanwhile, if NOT-IN-CORE is in P, then so is CORE-EMPTINESS, because the former is the separation oracle problem for the latter (see also the discussion in [14]). Summarizing in a proposition:

PROPOSITION 1. *The* CORE-EMPTINESS*,* NOT-IN-CORE *and* MOST-VIOLATED-COALITION *problems have nondecreasing complexity. That is, if a problem is in P, so are the problems on its left; if a problem is NP-hard, so are the problems on its right.*[3]

Hence, the most difficult hardness results to obtain are those for CORE-EMPTINESS. Moreover, it appears even more difficult to prove that this problem is hard if we require the game to be superadditive, for the following reason. We recall that a natural certificate for CORE-EMPTINESS consists of the right values for $w(S)$ in the Bondareva-Shapley theorem (or at least a specification of which ones are positive). It is natural to attempt reductions to CORE-EMPTINESS where, if the answer is *yes*, there exists a subset S such that setting $w(S) = 1$ and $w(A \setminus S) = 1$ (and everything else to 0) is a certificate. Indeed, this is the case for the proof by Deng and Papadimitriou [7]. However, this approach cannot work in superadditive games, because there we would obtain $\sum_S w(S)v(S) = v(S) + v(A \setminus S) \leq v(A)$. More generally, any reduction where, if the answer is *yes*, there is a certificate where all the $w(S)$ take value either 0 or 1—and this would seem to be the natural approach if we rely on reductions that involve finding a subset or partition of the agents with some property—cannot work if the game is superadditive; we fundamentally need to consider fractional values for the $w(S)$, as we do in what follows.

[3]The hardness direction requires a Turing (Cook) reduction.

4. SMALL-ISSUES-LARGE-TEAM GAMES

We recall that one family of games that can be represented compactly consists of those games whose characteristic functions are the sum of multiple individual issues, each of which concerns only a constant number of agents [4]. In this section, we consider a slightly more general class of games that, in addition to the small issues, allow a single issue that has nonzero value only for the grand coalition. This value may represent various efficiencies that could result from having all agents in the same coalition. We call these games *Small-Issues-Large-Team (SILT)* games. Besides being of interest in their own right, these games will help us prove hardness results for MC-nets. Specifically, we show that CORE-EMPTINESS is NP-hard in SILT games, even under the constraint of superadditivity (which also implies monotonicity because all the values are nonnegative). We then (in Section 5.1) show that some (though not all) important subclasses of MC-nets can compactly represent all SILT games, thereby proving hardness for them as well.

DEFINITION 5 (SILT GAME). *A* SILT *game is defined by a triplet (A, T, g_A) where A is the set of agents, T is the set of small issues, and g_A is the marginal contribution of the grand coalition. A small issue $t \in T$ is described by (v_t, C_t): a relevant agent set $C_t \subseteq A$ (where $|C_t|$ is bounded by a constant) and a characteristic function $v_t : 2^{C_t} \to \mathbb{R}$. v_t is extended to the domain 2^A by letting $v_t(S) = v_t(S \cap C_t)$. Furthermore there is the single large issue whose characteristic function $g : 2^A \to \mathbb{R}$ is given by $g(A) = g_A$ and $g(\cdot)$ is zero everywhere else. The game's characteristic function overall $v : 2^A \to \mathbb{R}$ is given by $v(S) = g(S) + \sum_{t \in T} v_t(S \cap C_t)$.*

Next, we give a reduction to SILT games from the NP-complete VERTEX-COVER problem, in which we are given a graph G and a number k and are asked whether there exists a subset of at most k vertices such that every edge includes a vertex in the subset.

DEFINITION 6 (VERTEX-COVER SILT GAME). *Let (G, k) be a* VERTEX-COVER *instance, where $G = (V, E)$ and $|V| = n$. We define the corresponding Vertex-Cover SILT game (A, T, g_A) as follows. The agent set is $A = A_0 \cup V$ where $A_0 = \{a_0, a_1, \ldots, a_4\}$ is a set of 5 auxiliary agents. The set of issues is $T = V \cup E$. A vertex issue $i \in V \subseteq T$ concerns only agents in $C_i = A_0 \cup \{i\}$ and an edge issue $e = \{i, j\} \in E \subseteq T$ concerns only agents in $C_e = A_0 \cup \{i, j\}$.*

For a coalition S and a vertex issue $i \in V$, $v_i(S) = \frac{1}{n(n+1)}$ if $i \wedge [(a_1 \wedge a_2 \wedge ((\neg a_3 \wedge \neg a_4) \vee \neg a_0)) \vee (a_3 \wedge a_4 \wedge ((\neg a_1 \wedge \neg a_2) \vee \neg a_0))]$ and $v_i(S) = 0$ otherwise. Here, a positive literal z in the boolean expression means that agent $z \in S$, and a negative literal $\neg z$ means $z \notin S$. Similarly, for a coalition S and an edge issue $e = \{i, j\}$, $v_e(S) = 1$ if $(i \vee j) \wedge a_0 \wedge ((a_1 \wedge a_3) \vee (a_2 \wedge a_4))$ and $v_e(S) = 0$ otherwise.

Intuitively, if we arrange the 5 auxiliary agents as in Figure 1, a vertex issue i contributes non-zero value if and only if (1) $i \in S$, (2) no diagonal $\{a_1, a_0, a_3\}$ or $\{a_2, a_0, a_4\}$ is completely included in S, and (3) either the left column $\{a_1, a_2\}$ or the right column $\{a_3, a_4\}$ (or both) is completely included in S. An edge issue $e = \{i, j\}$ contributes non-zero value if and only if (1) $\{i, j\}$ intersects with S and (2) either diagonal $\{a_1, a_0, a_3\}$ or diagonal $\{a_2, a_0, a_4\}$ (or both) is included in S.

Finally, the grand coalition's marginal contribution is $g_A = \frac{n-k}{n(n+1)} \times \frac{2n+1}{2n+2}$.

LEMMA 2. *The* VERTEX-COVER *instance (G, k) has* yes *as its answer if and only if its corresponding* VERTEX-COVER *SILT game in Definition 6 has an empty core.*

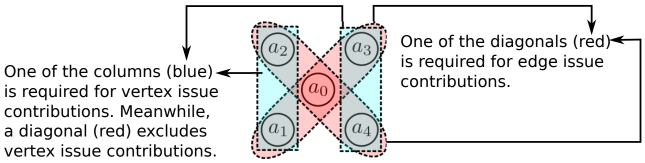

Figure 1: How auxiliary agents in Definition 6 determine whether an issue contributes non-zero value or not.

PROOF. First, we show that the existence of a vertex cover of size at most k implies that the core is empty. We can handle this direction quite easily thanks to the Bondareva-Shapley Theorem. Let $S \subseteq V$ be a vertex cover of size at most k. We construct the following weight function $w : 2^A \to [0, 1]$:

$$w(\{a_0, a_1, a_3\} \cup S) = w(\{a_0, a_2, a_4\} \cup S) = 1/2$$
$$w(\{a_1, a_2\} \cup (V \setminus S)) = w(\{a_3, a_4\} \cup (V \setminus S)) = 1/2$$

and $w = 0$ everywhere else. It is straightforward to check that for all $a \in A$, $\sum_{S\subseteq A: a\in S} w(S) \le 1$. We have $v(\{a_0, a_1, a_3\} \cup S) = v(\{a_0, a_2, a_4\} \cup S) = |E|$: these coalitions will get all of the edge contributions (because S is a vertex cover), none of the vertex contributions (because they contain a diagonal), and they will not get g_A. Also, we have $v(\{a_1, a_2\} \cup (V \setminus S)) = v(\{a_3, a_4\} \cup (V \setminus S)) \ge \frac{n-k}{n(n+1)}$: these coalitions will get none of the edge contributions (because they do not contain a_0), at least $n - k$ of the vertex contributions (specifically, the contributions for all vertices $i \in V \setminus S$), and they will not get g_A. Finally, we have $v(A) = |E| + \frac{n-k}{n(n+1)} \times \frac{2n+1}{2n+2}$, because the grand coalition will get all of the edge contributions and g_A, but none of the vertex contributions. Hence, we can conclude that $\sum_{S\subseteq A} w(S)v(S) \ge |E| + \frac{n-k}{n(n+1)} > |E| + \frac{n-k}{n(n+1)} \times \frac{2n+1}{2n+2} = v(A)$. It follows that the conditions of the Bondareva-Shapley theorem are satisfied and the core is empty.

Next, we show that emptiness of the core implies that a vertex cover of size at most k exists; this is the more difficult direction. For the sake of contradiction, let us assume that the core is empty and every vertex cover has size at least $k + 1$. The emptiness of the core implies the existence of a weight function $w : 2^A \to [0, 1]$ with the conditions specified in Theorem 1. Without loss of generality, we can assume $w(A) = 0$. (If $w(A) > 0$, then we can construct the modified weight function w' with $w'(A) = 0$ and $w'(S) = w(S)/(1 - w(A))$ when $S \ne A$. Then, for any $a \in A$, $\sum_{S\subseteq A: a\in S} w'(S) = 1/(1 - w(A)) \sum_{S\subsetneq A: a\in S} w(S) \le (1/(1 - w(A)))(1 - w(A)) = 1$, where the inequality is due to the fact that w satisfies Inequality 2. Moreover, $\sum_{S\subseteq A} w'(S)v(S) = 1/(1 - w(A)) \sum_{S\subsetneq A} w(S)v(S) > (1/(1-w(A)))(v(A)-w(A)v(A)) = v(A)$, where the inequality is due to the fact that w satisfies Inequality 1.) So we can restrict our attention to coalitions $S \subsetneq A$.

Now, we categorize all subsets $S \subsetneq A$ into two families, namely $\mathcal{S}_E = \{S \subsetneq A \mid v(S) \ge 1\}$ and $\mathcal{S}_V = 2^A \setminus (\mathcal{S}_E \cup \{A\})$. Because all vertex issues combined contribute strictly less than 1 and the grand coalition marginal contribution will not occur as we consider only coalitions $S \subsetneq A$, it follows that $\mathcal{S}_E$ is exactly the family of subsets S that obtain at least one contribution of 1 from an edge issue. Hence, $\mathcal{S}_V$ is the family of subsets S that only derive value from vertex issues. We can then rewrite Inequality 1 as

$$\sum_{S\in\mathcal{S}_E} w(S)v(S) + \sum_{S\in\mathcal{S}_V} w(S)v(S) > v(A) \tag{3}$$

Let $\mathcal{S}_E^* = \{S \in \mathcal{S}_E \mid v(S) \ge |E|\}$ consist of the coalitions that correspond to vertex covers, and let $p = \sum_{S\in\mathcal{S}_E^*} w(S)$ and $q = \sum_{S\in\mathcal{S}_E\setminus\mathcal{S}_E^*} w(S)$. Because $a_0 \in \bigcap_{S\in\mathcal{S}_E} S$ and Inequality 2 holds, we have $p + q = \sum_{S\in\mathcal{S}_E} w(S) \le \sum_{S\subseteq A: a_0\in S} w(S) \le 1$. Therefore, $q \le 1-p$. Additionally, because for all $S \in \mathcal{S}_E$, $v(S)$ is an integer no greater than $|E|$ (the presence of a_0 precludes vertex issue contributions), we derive

$$\begin{aligned}\sum_{S\in\mathcal{S}_E} w(S)v(S) &= \sum_{S\in\mathcal{S}_E^*} w(S)v(S) + \sum_{S\in\mathcal{S}_E\setminus\mathcal{S}_E^*} w(S)v(S)\\ &\le p|E| + q(|E| - 1)\\ &\le p|E| + (1 - p)(|E| - 1) = |E| - 1 + p\end{aligned} \tag{4}$$

This corresponds to the first term in the left-hand side of Inequality 3. For the second term, since for $S \in \mathcal{S}_V$, each vertex can contribute at most $\frac{1}{n(n+1)}$, we obtain

$$\sum_{S\in\mathcal{S}_V} w(S)v(S) \le \sum_{S\in\mathcal{S}_V} w(S)|S \cap V| \times \frac{1}{n(n+1)}$$
$$= \frac{1}{n(n+1)} \sum_{v\in V} \sum_{\substack{S\in\mathcal{S}_V:\\ v\in S}} w(S) \tag{5}$$
$$\le \frac{1}{n(n+1)} \sum_{v\in V} 1 = \frac{1}{n+1} \tag{6}$$

where the last inequality is due to Inequality 2.

Combining Inequalities 4 and 6 with Inequality 3, we obtain $|E| - 1 + p + \frac{1}{n+1} \ge \sum_{S\in\mathcal{S}_E} w(S)v(S) + \sum_{S\in\mathcal{S}_V} w(S)v(S) > v(A) > |E|$, where $v(A) > |E|$ because the grand coalition A obtains all edge issue contributions plus a positive grand coalition marginal contribution. From this it follows that $p > \frac{n}{n+1}$.

We now recall that for every $S \in \mathcal{S}_E^*$, $S \cap V$ must be a vertex cover in G. By the assumption we made to derive a contradiction, every vertex cover has size at least $k + 1$, so for all $S \in \mathcal{S}_E^*$, $|S \cap V| \ge k + 1$. This implies

$$\sum_{v\in V} \sum_{\substack{S\in\mathcal{S}_E^*:\\ v\in S}} w(S) = \sum_{S\in\mathcal{S}_E^*} w(S)|S \cap V|$$
$$\ge \sum_{S\in\mathcal{S}_E^*} w(S)(k + 1) = (k + 1)p > \frac{n(k + 1)}{n + 1} \tag{7}$$

Again, using Inequality 2, we obtain $n \ge \sum_{v\in V} \sum_{\substack{S\subsetneq A:\\ v\in S}} w(S) \ge \sum_{v\in V} \left(\sum_{\substack{S\in\mathcal{S}_E^*:\\ v\in S}} w(S) + \sum_{\substack{S\in\mathcal{S}_V:\\ v\in S}} w(S) \right)$, which is at least $\frac{n(k+1)}{n+1} + \sum_{v\in V} \sum_{\substack{S\in\mathcal{S}_V:\\ v\in S}} w(S)$ by Inequality 7. Hence,

$$\sum_{v\in V} \sum_{\substack{S\in\mathcal{S}_V:\\ v\in S}} w(S) \le n - \frac{n(k + 1)}{n + 1} \tag{8}$$

Combining Inequality 8 with Inequality 5 we get

$$\sum_{S\in\mathcal{S}_V} w(S)v(S) \le \frac{1}{n(n+1)} \left(n - \frac{n(k + 1)}{n + 1} \right)$$
$$= \frac{1}{n + 1} - \frac{k + 1}{(n + 1)^2} = \frac{n - k}{(n + 1)^2} \tag{9}$$

Finally, using Inequality 9 and 4 together with $p \le 1$, we get $\sum_{S\in\mathcal{S}_E} w(S)v(S) + \sum_{S\in\mathcal{S}_V} w(S)v(S) \le |E| - 1 + p + \frac{n-k}{(n+1)^2} \le$

$|E| + \frac{n-k}{(n+1)^2} \leq |E| + \frac{n-k}{n(n+1)} \times \frac{2n+1}{2n+2} = v(A)$, which contradicts Inequality 3. $\square$

So far, we have said nothing about superadditivity. It can be checked that the VERTEX-COVER SILT game is superadditive when the graph G has no isolated vertex (which of course we can assume without affecting the hardness of VERTEX-COVER). However, some of the individual issues are not superadditive—in fact, they are not even monotone. Specifically, the value of a vertex issue may decrease to 0 if a_0 is added. We would like the hardness result to hold even under the stronger condition that every issue is required to be superadditive (and therefore, given that all values are nonnegative, also monotone). We next show that we can rearrange the issues in VERTEX-COVER SILT games so that the game remains the same but each issue is superadditive.

LEMMA 3. *If graph G has no isolated vertex, then the corresponding* VERTEX-COVER *SILT game can be rewritten as an equivalent SILT game each of whose individual issues is superadditive, monotone and nonnegative. This also implies that the whole game is superadditive, monotone and nonnegative.*

PROOF. Recall that the original issue set is $T = E \cup V$. We now construct a new set of issues $T' = E$ all of which are superadditive, monotone and nonnegative. We leave g_A unchanged and prove the equivalence of T and T' (the overall value function is the same).

For each new issue $e = \{i, j\} \in E = T'$, let its relevant agent set be $C_e = A_0 \cup \{i, j\}$. Let $d(i)$ denote the degree of vertex i in G. Let VCondition $= [(a_1 \wedge a_2 \wedge ((\neg a_3 \wedge \neg a_4) \vee \neg a_0)) \vee (a_3 \wedge a_4 \wedge ((\neg a_1 \wedge \neg a_2) \vee \neg a_0))]$, i.e., the condition on auxiliary agents to allow vertex issue contributions in the original VERTEX-COVER SILT game. Similarly, let ECondition $= a_0 \wedge ((a_1 \wedge a_3) \vee (a_2 \wedge a_4))$, i.e., the condition to allow edge issue contributions. We recall that VCondition and ECondition cannot hold simultaneously. For a coalition $S \subseteq A$, the new issue's contribution is $v'_e(S) = 1$ if $(i \vee j) \wedge$ ECondition, $v'_e(S) = \frac{1/d(i) + 1/d(j)}{n(n+1)}$ if $i \wedge j \wedge$ VCondition, $v'_e(S) = \frac{1/d(i)}{n(n+1)}$ if $i \wedge \neg j \wedge$ VCondition, and $v'_e(S) = \frac{1/d(j)}{n(n+1)}$ if $\neg i \wedge j \wedge$ VCondition. Otherwise, $v'_e(S) = 0$. To see why this is equivalent to the original game, note that we have amortized vertex i's issue contribution $\frac{1}{n(n+1)}$ over all of its adjacent edges, each with a partial contribution $\frac{1/d(i)}{n(n+1)}$. Because there is no isolated vertex, T and T' are equivalent.

It remains to prove superadditivity for each issue e's value function v'_e (which will imply monotonicity of v'_e as well by nonnegativity, and will imply superadditivity and monotonicity for the whole game too, because it is the sum of these issues and the grand coalition contribution which is also superadditive). Consider two disjoint coalitions $B, C \subseteq C_e$ where $B \cap C = \emptyset$. We prove $v'_e(B \cup C) \geq v'_e(B) + v'_e(C)$ by considering three cases. Without loss of generality, assume $v'_e(B) \geq v'_e(C)$.

(1) If $v'_e(B) = 0$ (and therefore $v'_e(C) = 0$), by nonnegativity $v'_e(B \cup C) \geq 0 = v'_e(B) + v'_e(C)$.

(2) If $v'_e(B) = 1$, then ECondition must be true for both B and $B \cup C$, while VCondition and ECondition must be both false for C (because B contains an entire diagonal and $B \cap C = \emptyset$, C can contain neither a_0 nor one of the columns). So $v'_e(B \cup C) = 1$ and $v'_e(C) = 0$, and we have $v'_e(B \cup C) = 1 \geq v'_e(B) + v'_e(C) = 1$.

(3) Finally, if $0 < v'_e(B) < 1$, VCondition must be true for B. If VCondition is also true for $B \cup C$, then $v'_e(B \cup C) \geq v'_e(B) + v'_e(C)$ because (1) C cannot make ECondition true as it does not include a diagonal, and (2) the amortized vertex contributions that result from B and C correspond to disjoint sets of vertices. On the other hand, if VCondition is false for $B \cup C$, then $B \cup C$ includes a diagonal and hence its ECondition is true. Thus $v'_e(B \cup C) = 1 \geq v'_e(B) + v'_e(C)$ (because B and C can only have vertex contributions from disjoint sets of vertices). $\square$

From Lemma 2, Lemma 3, and the fact that VERTEX-COVER is NP-hard (even when restricted to instances without isolated vertices), we obtain the following theorem.

THEOREM 2. *The* CORE-EMPTINESS *problem is NP-hard for SILT games, even when the value function for every issue is superadditive, monotone, and nonnegative (and hence the game's overall value function is also superadditive, monotone, and nonnegative).*

5. MARGINAL CONTRIBUTION NETS

A Marginal Contribution Net (MC-net) [14] represents a cooperative game using a set of patterns $\mathcal{P}$; furthermore, each pattern $P \in \mathcal{P}$ is associated with a value v_P. A pattern P is a boolean expression whose truth value depends on the coalition $S \subseteq A$. Its variables correspond to the agents; a will be shorthand for a being in the coalition. Denote by $P(S)$ the truth value of P on coalition S. The value of coalition S is then $v(S) = \sum_{P \in \mathcal{P}: P(S)=\textit{true}} v_P$. For example, we may have one pattern $(a_1 \wedge \neg a_2)$ with value 1 and another pattern (a_2) with value 2. This implies $v(\{a_1\}) = 1 + 0 = 1$ and $v(\{a_1, a_2\}) = 0 + 2 = 2$.

We consider a variety of subclasses of MC-nets, which are defined by whether they allow the use of: (1) negative pattern values, (2) the $\wedge$ operator, (3) the $\vee$ operator, and (4) the $\neg$ operator. By enumerating all possible combinations of these four attributes, we define 16 subclasses of MC-nets, as shown in Table 1. (A possible fifth attribute is whether brackets are allowed; fortunately, as we discuss in Remark 1, that attribute is irrelavant to the computational complexity of our problems, except in one case.)

To enumerate the subclasses, we write four binary indicators consecutively (corresponding to the four attributes), and interpret the result as a binary number. For example, Class 0 (0000 in binary) MC-nets have all indicators equal to 0, which means this subclass allows neither negative pattern values nor the use of any of the operators $\wedge$, $\vee$, and $\neg$. Hence, in Class 0, each pattern is either empty (*true*) or consists of one agent a only. Class 13 (1101 in binary) MC-nets can have negative pattern values as well as $\wedge$ and $\neg$ operators, but not $\vee$ operators. These are the original MC-nets introduced by [14]. The complexity results obtained in this section are summarized in Table 1.

5.1 Classes Where All Problems Are Unambiguously Hard

In this subsection, we will prove the following theorem, which shows that all three problems are NP-complete for a number of subclasses of MC-nets. It is based on several lemmas that will be stated and proved afterwards. There are two other subclasses for which we will prove in later subsections that all problems are hard *if* we make some further assumptions, namely the ability to specify the grand coalition's value in one case (Section 5.2), and the ability to use brackets in the other case (Remark 1). All of these results are by reduction from SILT games. For other classes, the CORE-EMPTINESS problem is in P, and so SILT games cannot be reduced to them (unless P=NP).

THEOREM 3. *All three problems,* CORE-EMPTINESS, NOT-IN-CORE, *and* MOST-VIOLATED-COALITION *are NP-complete for Class 5, 6, 7, 11, 12, 13, 14, and 15 MC-nets.*[4]

[4]Again, our general reduction from CORE-EMPTINESS to NOT-IN-CORE and MOST-VIOLATED-COALITION is a Turing reduc-

Class	$-$	$\wedge$	$\vee$	$\neg$	CORE-EMPTINESS	NOT-IN-CORE	MOST-VIOLATED-COALITION
0	0	0	0	0	P	P	P
1	0	0	0	1	P	P	P
2	0	0	1	0	P	P	NP-c
3	0	0	1	1	P†	P†	NP-c
4	0	1	0	0	P	P	P
5	0	1	0	1	NP-c	NP-c	NP-c
6	0	1	1	0	NP-c	NP-c	NP-c
7	0	1	1	1	NP-c	NP-c	NP-c
8	1	0	0	0	P	P	P
9	1	0	0	1	P	P	P
10	1	0	1	0	NP-c*	NP-c	NP-c
11	1	0	1	1	NP-c	NP-c	NP-c
12	1	1	0	0	NP-c	NP-c	NP-c
13	1	1	0	1	NP-c	NP-c	NP-c
14	1	1	1	0	NP-c	NP-c	NP-c
15	1	1	1	1	NP-c	NP-c	NP-c

Table 1: Complexity results for MC-nets. The four binary indicators correspond to whether (1) negative values, (2) $\wedge$ operators, (3) $\vee$ operators, and (4) $\neg$ operators are allowed. For entries with NP-c*, we only proved NP-hardness when we are allowed to specify the grand coalition value directly. For entries with P†, the result changes to NP-c if brackets are allowed.

PROOF. From Lemma 4, it will follow that CORE-EMPTINESS is NP-hard for Class 5 MC-nets. This hardness extends to Class 7 MC-nets, which are more expressive. From Lemma 5, it will follow that CORE-EMPTINESS is NP-hard for Class 12 MC-nets, which extends to Class 13, 14, and 15 MC-nets. From Lemma 6 and Lemma 7 respectively, it will follow that CORE-EMPTINESS is NP-hard for Class 11 and 6 MC-nets, respectively. By Proposition 1 this also implies NP-hardness for the other two problems. Finally, by Lemma 1, all problems are in NP. □

LEMMA 4 (CLASS 5 MC-NETS). *Any SILT game with nonnegative issues can be represented in polynomial size by an MC-net that uses neither negative pattern values nor the $\vee$ operator (but it may use the $\wedge$ and $\neg$ operators).*

PROOF. First, construct a pattern $\bigwedge_{a\in A} a$ with value g_A to represent the grand coalition marginal contribution. Then, for each issue t with relevant agent set C_t, we add $2^{|C_t|}$ (which is a constant number) patterns, as follows. For each subset $S \subseteq C_t$, we add the pattern $P_S = \bigwedge_{a\in S} a \wedge \bigwedge_{a\notin S} \neg a$ with value $v_t(S)$. □

LEMMA 5 (CLASS 12 MC-NETS). *Any SILT game can be represented in polynomial size by an MC-net that uses neither the $\vee$ operator nor the $\neg$ operator (but it may use the $\wedge$ operator and negative pattern values).*

PROOF. Again, we construct a pattern $\bigwedge_{a\in A} a$ with value g_A. For each issue t with relevant agent set C_t, we add $2^{|C_t|}$ patterns, as follows. For each subset $S \subseteq C_t$, in order of nondecreasing size, we add the pattern $P_S = \bigwedge_{a\in S} a$. We determine its value as follows. We know that for the coalition S, this pattern will apply; also, we have already specified the values for all the other patterns that will apply, because these involve coalitions that are subsets of S. Let $v_t^{\text{partial}}(S)$ denote the sum of the values of these previously specified patterns. Then, set the value of the new pattern to $v_t(S) - v_t^{\text{partial}}(S)$, thereby guaranteeing that S obtains the correct value. (In fact, the value for the pattern P_S will turn out to be $\sum_{S'\subseteq S}(-1)^{|S|-|S'|}v_t(S')$. See, e.g., [3].) □

tion. If it is desired to avoid Turing reductions, it can be noted that [4] already showed that NOT-IN-CORE is NP-hard for SILT games, even when the grand coalition's marginal contribution is zero (and hence so is MOST-VIOLATED-COALITION).

LEMMA 6 (CLASS 11 MC-NETS). *Any SILT game can be represented in polynomial size by an MC-net that does not use the $\wedge$ operator (but it may use the $\vee$ and $\neg$ operators as well as negative pattern values).*

PROOF. By Lemma 5, we can represent any SILT game in polynomial size by an MC-net that uses only the $\wedge$ operator (and possibly negative values). Let the set of patterns of that MC-net be $\mathcal{P}$. In what follows, we construct a new MC-net with pattern set $\mathcal{P}'$ that has only $\vee, \neg$ operators (and possibly negative values), and show that $\mathcal{P}$ and $\mathcal{P}'$ result in the same value for each coalition.

For the jth pattern $P_j = z_1 \wedge z_2 \wedge \ldots \wedge z_m \in \mathcal{P}$ with value v_j, we add a pattern $P'_j = \neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_m$ with value $v'_j = -v_j$ to $\mathcal{P}'$. P'_j evaluates to *true* if and only if P_j evaluates to *false*. Finally, we add one additional dummy pattern $P'_0 = a \vee \neg a$ for some arbitrary $a \in A$ (or, equivalently, simply $P'_0 = \textit{true}$), with value $v(A)$, i.e., the value of the grand coalition (not just its marginal contribution). The dummy pattern always contributes value $v(A)$ to any coalition S. Now we check that for any coalition S, its value $v'(S)$ under the new patterns P' is equal to its value $v(S)$ under the old patterns P. Let $J = \{j \mid P_j \text{ is true for } S\}$ and $J' = \{j \mid P'_j \text{ is true for } S\}$. Then $v(S) = \sum_{j\in J} v_j = \sum_{j=1}^{|\mathcal{P}|} v_j - \sum_{j\notin J} v_j = v(A) - \sum_{j\notin J} v_j = \sum_{j\in J'} v'_j = v'(S)$. □

LEMMA 7 (CLASS 6 MC-NETS). *Any SILT game with nonnegative and monotone issues can be represented in polynomial size by an MC-net that uses neither negative pattern values nor the $\neg$ operator (but it may use the $\wedge$ and $\vee$ operators).*

PROOF. For any such SILT game, we construct an MC-net with patterns $\mathcal{P} = \bigcup_{t\in T} \mathcal{P}_t \cup \{P_g\}$, where $P_g = \bigwedge_{a\in A} a$ with value g_A represents the grand coalition's marginal contribution. $\mathcal{P}_t$ is a set of patterns for issue t that only concerns agents in C_t.

All that remains to be done is to show how to construct $\mathcal{P}_t$ and to prove its equivalence to issue t. Let $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m\}$ be a partition of 2^{C_t} based on v_t. That is, there exists a strictly increasing sequence of values $v^1 < v^2 < \ldots < v^m$ such that for all $S \in \mathcal{S}_i$, $v_t(S) = v^i$. Define $\mathcal{S}_i^+ = \bigcup_{j=i}^m S_j$. We then construct $\mathcal{P}_t = \{P_1, P_2, \ldots, P_m\}$ where $P_i = \bigvee_{S\in\mathcal{S}_i^+} \bigwedge_{a\in S} a$ has value $v^i - v^{i-1}$, defining $v^0 = 0$. (We note that this expression does not require brackets because $\wedge$ has higher priority than $\vee$.)

Since $v^1 \geq 0$ (due to nonnegativity) and by construction $v^i < v^{i+1}$, all pattern values $v^i - v^{i-1}$ are nonnegative. For any subset $S \subseteq C_t$, there exists some i such that $S \in \mathcal{S}_i$. For any $j \leq i$, S will satisfy P_j because $S \in \mathcal{S}_i \subseteq \mathcal{S}_j^+$. We next show that for any $j > i$, S does not satisfy P_j. For the sake of contradiction, suppose that S does satisfy some P_j with $j > i$. Then, there exists $S' \in \mathcal{S}_j^+$ such that $S' \subseteq S$. However, because $i < j$, it follows that $v_t(S) < v_t(S')$. This contradicts the monotonicity of issue t. Hence, $S \in \mathcal{S}_i$ satisfies P_j if and only if $j \leq i$. Therefore, S's value under pattern $\mathcal{P}_t$ equals $\sum_{j=1}^i v^j - v^{j-1} = v^i = v_t(S)$. □

5.2 MC-nets with Only OR Operators

If only the $\vee$ operator is available, it is impossible for MC-nets to compactly represent the grand coalition's marginal contribution g_A. For example, suppose $v(S) = 0$ for any $S \neq A$ and $v(A) = g_A = 1$. The $\vee$-only MC-net (with possibly negative pattern values) to specify this game uses $2^n - 1$ patterns: for each nonempty subset $\emptyset \neq S \subseteq A$, construct a pattern $\bigvee_{a \in S} a$ with value $(-1)^{|S|-1}$ [21]. Hence, we cannot hope to compactly represent SILT games with this subclass of MC-nets; we need to add some additional representational power to do so. The minimal additional representational power that will do the trick is to simply allow the direct specification of the value of grand coalition, so this is what we do here. At the end of the paper, we discuss the open problem of determining the complexity of CORE-EMPTINESS for SILT games without a marginal contribution for the grand coalition (i.e., games consisting only of small issues). If this problem is still NP-hard, then we could drop the additional power in this subsection, because we do not need it to represent the small issues. In fact, [4] already showed that NOT-IN-CORE is NP-hard for such games (and hence so is MOST-VIOLATED-COALITION). So for these problems we do not need the additional power. This results in the following theorem:

THEOREM 4. NOT-IN-CORE *and* MOST-VIOLATED-COALITION *are NP-complete for Class* 10 *MC-nets.* CORE-EMPTINESS *is also NP-complete for Class* 10 *MC-nets* if *they are given the additional power to specify the value of the grand coalition.*

The theorem is implied by the following lemma.

LEMMA 8 (CLASS 10 MC-NETS). *Any SILT game can be represented in polynomial size by an MC-net that uses neither the $\wedge$ operator nor the $\neg$ operator (but it may use the $\vee$ operator and negative pattern values, as well as the additional power to specify the value of the grand coalition in case $g_A \neq 0$).*

PROOF. For each issue t with relevant agent set C_t, we add a set $\mathcal{P}_t$ of $2^{|C_t|}$ patterns, as follows. First, we add the empty pattern $P_\emptyset = \mathit{true}$, with value $v_t(\emptyset)$ (which in most circumstances would be 0). For each subset $S \subseteq C_t$ ($S \neq \emptyset$), in order of nondecreasing size, we add the pattern $P_S = \bigvee_{a \in S} a$ to $\mathcal{P}_t$. We determine its value as follows. Our goal is to ensure that if we apply $\mathcal{P}_t$ to C_t, as well as to $C_t \setminus S$, this results in two values whose difference is exactly $d_t(S) = v_t(C_t) - v_t(C_t \setminus S)$. ($d_t$ is also known as the dual of v_t [10].) C_t will satisfy every pattern; the only patterns that are not satisfied by $C_t \setminus S$ concern only agents in S. One such pattern is $\bigvee_{a \in S} a$, whose value we are seeking to specify now; for all the other such patterns, we have already specified their values. Let $d_t^{\text{partial}}(S)$ denote the sum of the values of these previously specified patterns. Then, set the value of the new pattern to $d_t(S) - d_t^{\text{partial}}(S)$, thereby guaranteeing that the difference in values is correct. (In fact, when $v_t(\emptyset) = 0$, the value for the pattern P_S will turn out to be $\sum_{S' \subseteq S} (-1)^{|S|-|S'|+1} v_t(C_t \setminus S')$, by an inclusion-exclusion principle.)

With these patterns, we obtain a value function v'_t. By construction, for any $\emptyset \neq S \subseteq C_t$, we have $v'_t(C_t) - v'_t(C_t \setminus S) = v_t(C_t) - v_t(C_t \setminus S)$. Moreover, $v'_t(\emptyset) = v_t(\emptyset)$. Therefore, $v'_t(C_t) = v'_t(C_t \setminus C_t) + v_t(C_t) - v_t(C_t \setminus C_t) = v'_t(\emptyset) + v_t(C_t) - v_t(\emptyset) = v_t(C_t)$. Hence, also for all $S \subsetneq C_t$, $v'_t(S) = v'_t(C_t \setminus (C_t \setminus S)) = v_t(C_t \setminus (C_t \setminus S)) + v'_t(C_t) - v_t(C_t) = v_t(S)$. Hence, the patterns correctly represent the issue.

Finally, if $g_A \neq 0$, we can use the ability to specify the grand coalition value to represent this. $\square$

5.3 Classes with Some Easy Problems

Note that there is no superadditivity constraint in what follows.

LEMMA 9. *The* MOST-VIOLATED-COALITION *problem is in P for MC-nets without the $\vee$ and $\wedge$ operators (but possibly using the $\neg$ operator and negative pattern values).*

PROOF. Let $A = \{a_1, a_2, \ldots, a_n\}$. Besides (possibly) the empty pattern, there are at most $2n$ distinct patterns, namely $a_1, a_2, \ldots, a_n$ and $\neg a_1, \neg a_2, \ldots, \neg a_n$. Let their respective values be $b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_n$ (possibly 0). The most violated coalition is $S = \{a_i \mid b_i - c_i - x_i > 0\}$, which can be computed in linear time. $\square$

Therefore, by Proposition 1, we conclude:

THEOREM 5. *All three problems,* CORE-EMPTINESS, NOT-IN-CORE *and* MOST-VIOLATED-COALITION*, are in P for Class* 0, 1, 8 *and* 9 *MC-nets.*

We now consider MC-nets that use neither $\wedge$ nor negative values (but $\vee$ and $\neg$ may occur). We show that this necessarily results in *subadditive* games. (A game is *subadditive* if $S_1 \cap S_2 = \emptyset$ implies that $v(S_1 \cup S_2) \leq v(S_1) + v(S_2)$.) From that, we prove that their NOT-IN-CORE and CORE-EMPTINESS problems are easy. However, the MOST-VIOLATED-COALITION problem is still hard.

LEMMA 10. *MC-nets that use neither $\wedge$ nor negative values are subadditive.*

PROOF. First consider an MC-net with just a single pattern, so that every coalition has value v_P or 0. Let the pattern be $P = \vee_{a \in S_P^+} a \vee_{a \in S_P^-} \neg a$. Then, subadditivity could only be violated if there are some S_1, S_2 with $S_1 \cap S_2 = \emptyset$, $v(S_1) = 0$, $v(S_2) = 0$, but $v(S_1 \cup S_2) = v_P$. But this means either (1) that there exists $a \in (S_1 \cup S_2) \cap S_P^+$, in which case either $a \in S_1 \cap S_P^+$ or $a \in S_2 \cap S_P^+$, contradicting that $v(S_1) = 0$ and $v(S_2) = 0$; or (2) that there exists $a \in S_P^- \setminus (S_1 \cup S_2)$, in which case also $a \in S_P^- \setminus S_1$, contradicting that $v(S_1) = 0$. Hence, subadditivity holds if there is only a single pattern. If there are multiple patterns, the resulting game is the sum of single-pattern games, which is also subadditive. $\square$

LEMMA 11. *If a game is subadditive and the value of each singleton coalition $v(\{a\})$ can be computed in polynomial time, then the* NOT-IN-CORE *problem is in P.*

PROOF. We will show that it is sufficient to check only singleton subsets $\{a\}$. If for some $a \in A$, $v(\{a\}) > x(a)$, x is not in the core. Otherwise, if $x(a) \geq v(a)$ for all $a \in A$, then x must be in the core because by subadditivity, $v(S) \leq \sum_{a \in S} v(\{a\}) \leq \sum_{a \in S} x(a) = x(S)$. $\square$

LEMMA 12. *The* MOST-VIOLATED-COALITION *problem is NP-hard for MC-nets with neither the $\wedge$ operator nor negative values (but possibly using the $\vee$ and $\neg$ operators).*

PROOF. We reduce from an arbitrary HITTING-SET instance, in which we are given m subsets $H_1, H_2, \ldots, H_m \subseteq N$ with $N = \{1, \ldots, n\}$ and a number $k \leq n$, and are asked whether there is a subset $H \subseteq N$ with $|H| \leq k$ such that for all $1 \leq i \leq m$, $H \cap H_i \neq \emptyset$. We construct an MC-net with $n+1$ agents $(a_0, a_1, \ldots, a_n)$ and m patterns P_1 to P_m, where $P_i = \bigvee_{j \in H_i} a_j$ has value 1, and consider payment vector x with $x(a_0) = m - n/(n+1)$ and for $1 \leq j \leq n$, $x(a_j) = 1/(n+1)$. Note that a_0 is a dummy agent whose purpose is to ensure $x(A) = v(A)$. We ask whether a coalition S exists such that $v(S) - x(S) > m - (k+1/2)/(n+1)$. If a hitting set H of size at most k exists, then the coalition $S_H = \{a_j | j \in H\}$ has $v(S_H) = m$ and $x(S_H) \leq k/(n+1) < (k+1/2)/(n+1)$, so the answer to our MOST-VIOLATED-COALITION is *yes*. Conversely, if a coalition S with $v(S) - x(S) > m - (k+1/2)/(n+1)$ exists, then it must satisfy all patterns, because otherwise $v(S) - x(S) \leq m - 1 < m - (k+1/2)/(n+1)$, and we must have $|S| \leq k$, because otherwise $v(S) - x(S) \leq m - (k+1)/(n+1) < m - (k+1/2)/(n+1)$. Therefore $H_S = \{j | a_j \in S\}$ is a hitting set of size at most k. $\square$

THEOREM 6. *For Class* 2 *and* 3 *MC-nets, the* CORE-EMPTINESS *and* NOT-IN-CORE *problems are in P while the* MOST-VIOLATED-COALITION *problem is NP-complete.*

PROOF. By Lemma 10, these MC-nets are subadditive, so by Lemma 11 and Proposition 1, CORE-EMPTINESS and NOT-IN-CORE are in P. On the other hand, MOST-VIOLATED-COALITION is NP-complete by Lemma 1 and 12. □

Finally, class 4 MC-nets, in which neither $\vee$, $\neg$, nor negative pattern values are used, are equivalent to hypergraph games with only nonnegative edge values, which were studied in [7] and proved to be easy (see also Section 2). We omit the proof to save space.

THEOREM 7. CORE-EMPTINESS*,* NOT-IN-CORE*, and* MOST-VIOLATED-COALITION *are in P for Class* 4 *MC-nets.*

REMARK 1. *So far, we have not considered the possibility of using brackets in the patterns of MC-nets; here we do so. Because this will only make them harder to solve, all NP-completeness results still hold. Therefore, let us consider the subfamilies of MC-nets for which some problems are in P.*

For MC-nets without connectives (Class 0, 1, 8, 9), MC-nets with only $\wedge$ operators (Class 4), and MC-nets with only $\vee$ operators (Class 2), brackets make no difference to the patterns. For MC-nets with both $\vee$ and $\neg$ operators (Class 3), adding brackets allows them to simulate $P_1 \wedge P_2$ by $\neg(\neg P_1 \vee \neg P_2)$. Hence they can encode Class 7 MC-nets, implying NP-completeness for all three problems.

6. CONCLUSION

We settled the complexity of the CORE-EMPTINESS, NOT-IN-CORE, and MOST-VIOLATED-COALITION problems in several subclasses of MC-nets, defined by which logical operators and whether negative pattern values are allowed (see Table 1). (Efficient algorithms for the MOST-VIOLATED-COALITION problem also allow efficient computation of the least core and, under certain conditions, the nucleolus [9].) To obtain these results, we introduced SILT games, which may be of interest in their own right, and proved hardness for them even under the constraint that each of their issues is superadditive. We showed certain subclasses of MC-nets can efficiently represent those hard SILT games. (We suspect that many other representation schemes that are not necessarily subclasses of MC-nets would be able to efficiently represent these SILT games, and hence our hardness results apply to such schemes as well.) For other subclasses of MC-nets, we directly proved results.

Different from previous hardness results, all our hardness results based on SILT games hold even with superadditivity, an extremely common constraint for cooperative games. Achieving this for the CORE-EMPTINESS problem necessarily (as argued at the end of Section 3.2) requires an involved reduction.

Our results leave open whether the hardness we proved for SILT games would continue to hold even when the marginal contribution of the grand coalition is zero (so that the game consists only of small issues [4]). If this is so, it would give us an even more powerful result that would allow us to prove hardness for even more representation schemes. As just one example, we would cleanly obtain hardness for Class 10 MC-nets, rather than just for the extension of this subclass that additionally allows specifying $v(A)$.

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