Ex post Efficiency of Random Assignments

(Extended Abstract)

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ABSTRACT

In the random assignment problem, objects are randomly assigned to agents keeping in view the agents' preferences over objects. A random assignment specifies the probability of an agent getting an object. We examine the structural and computational aspects of ex post efficiency of random assignments. We first show that whereas an ex post efficient assignment can be computed easily, checking whether a given random assignment is ex post efficient is NP-complete. Hence implementing a given random assignment via deterministic Pareto optimal assignments is NP-hard. We then formalize another concept of efficiency called robust ex post efficiency that is weaker than stochastic dominance efficiency but stronger than ex post efficiency. We present a characterization of robust ex post efficiency and show that it can be tested in polynomial time if there are a constant number of agent types (with identical preferences). Finally, we show that whereas robust ex post efficiency depends solely on which entries of the assignment matrix are zero/non-zero, ex post efficiency of an assignment depends on the actual values.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences - Economics

General Terms

Economics, Theory and Algorithms

Keywords

Resource allocation; efficiency; Pareto optimality

1. INTRODUCTION

Pareto optimality has been termed the *"single most important tool of normative economic analysis"* [12]. It appeals to the idea that there should not exist another possible outcome different from the social outcome which all the agents prefer. We consider Pareto optimality in the *random assignment problem* which is a fundamental and widely applicable setting in computer science and economics (see e.g., [13, 9, 11, 7, 10, 3]).

Random assignment problem.

In a *random assignment problem* $(N, O, \succsim)$, there is a set of agents $N = \{1, \ldots, n\}$, a set of objects $O = \{o_1, \ldots, o_n\}$, and a preference profile $\succsim = (\succsim_1, \ldots, \succsim_n)$ that specifies for each agent $i \in N$ his *strict*, complete and transitive preferences $\succsim_i$ over objects in O. A *random assignment* which we will simply refer to as assignment assigns the probability of agents getting objects. It is represented by a $(n \times n)$ matrix $[p(i)(o_j)]$ such that $p(i)(o_j) \in [0, 1]$ for all $i \in N$, and $o_j \in O$, ; $\sum_{i \in N} p(i)(o_j) = 1$ for all $o_j \in O$; and $\sum_{o_j \in O} p(i)(o_j) = 1$ for all $i \in N$. The value $p(i)(o_j)$ represents the probability of object o_j being allocated to agent i. Each row $p(i) = (p(i)(o_1), \ldots, p(i)(o_n))$ represents the allocation of agent i. The set of columns correspond to the objects $o_1, \ldots, o_n$. A feasible random assignment is *deterministic* if $p(i)(o) \in \{0, 1\}$ for all $i \in N$ and $o \in O$. It is well-known that any random assignment can be a result of a probability distribution over deterministic assignments [8]. When a random assignment p is represented as a convex combination of deterministic assignments, we will refer to the convex combination as a *decomposition* of p. We say that a deterministic assignment q is *consistent* with a random assignment p if for each $p(i)(o) = 1$, we have that $p(i)(o) > 0$.

Efficiency notions.

In this paper, we focus on efficiency of random assignments. A deterministic assignment p is *Pareto optimal* if there exists no other deterministic assignment q such that each agent weakly prefers his object allocated in assignment q and at least one agent strictly prefers his object allocated in assignment q. When the assignment is random, Pareto optimality can be generalized to two well-studied efficiency concepts — ex post efficiency and stochastic dominance (SD) efficiency. A random assignment is *ex post efficient* if it can be represented as a convex combination of Pareto opti-

mal deterministic assignments. A random assignment is SD-efficient is there exists no other random assignment which each agent weakly prefers and some agent strictly prefers with respect to the stochastic dominance relation. The SD relation is defined as follows. Given two random assignments p and q, $p(i) \succsim_i^{SD} q(i)$ i.e., an agent i *SD prefers* allocation $p(i)$ to allocation $q(i)$ if

$$\sum_{o_j \in \{o_k : o_k \succsim_i o\}} p(i)(o_j) \quad \geq \quad \sum_{o_j \in \{o_k : o_k \succsim_i o\}} q(i)(o_j) \quad \text{for all } o \in O.$$

Ex post efficiency is a weaker requirement than *stochastic dominance (SD) efficiency* [11]. It has been shown that not only can an SD-efficient random assignment be computed efficiently [9], a linear programming formulation can be used to check whether an assignment is SD-efficient or not [2]. However, to the best of our knowledge, the complexity of testing ex post efficiency has not been settled. The main research problem in this paper is to understand *the structure and complexity of efficient assignments in particular ex post efficient assignments.*

2. RESULTS

We first examine the problem of checking whether a given random assignment is ex post efficient and obtain insights into why the problem may be computationally challenging. We show that whereas computing an ex post efficient assignment is easy, checking whether a given random assignment is ex post efficient is NP-complete. Hence implementing a given random assignment via deterministic Pareto optimal assignments is NP-hard. Even if it is known that a random assignment is ex post efficient, finding its Pareto optimal decomposition is NP-hard. Our result also implies that optimizing over the convex hull of Pareto optimal assignments is NP-complete.

	Verification complexity	Combinatorial
Ex post	**NP-complete**	**no**
Robust ex post	**in coNP,** **in P for const # agent types**	**yes**
SD	in P (Theorem 1, [2])	yes (Lemma 3, [9])

Table 1: Summary of results and related work.

We formalize a new efficiency concept called *robust ex post efficiency* that is weaker than SD-efficiency but stronger than ex post efficiency. We say that a random assignment is *robust ex post efficient* if any decomposition of the assignment consists of Pareto optimal deterministic assignments. A characterization of robust ex post efficiency is also presented: An assignment is robust ex post efficient iff it does not admit a non-Pareto optimal deterministic assignment consistent with it. Previously, characterizing SD-efficiency has already attracted considerable interest (see e.g., [1, 2, 9]). On the computational front, we show that robust efficiency can be checked in polynomial time if there are a constant number of agent types.

We also check whether an efficiency concept is *combinatorial* or not. An efficiency concept X is *combinatorial* if for any two random assignments p and q such that $q(i)(o) > 0$ if and only if $p(i)(o) > 0$, it holds that p is efficient with respect to X if and only if q is efficient with respect to X. The notion of an efficiency concept being combinatorial was first discussed in [5] but in the context of voting. We show that whereas robust ex post efficiency is combinatorial, ex post efficiency is not. The finding that ex post efficiency is not combinatorial also contrasts with the fact that in randomized voting, ex post efficiency of a lottery simply depends on its support. Table 1 summarizes some of the results. Details about the results are available from [6].

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