Waste Makes Haste: Bounded Time Protocols for Envy-Free Cake Cutting with Free Disposal

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ABSTRACT

We consider the classic problem of envy-free division of a heterogeneous good (aka the cake) among multiple agents. It is well known that if each agent must receive a contiguous piece then there is no finite protocol for the problem, whenever there are 3 or more agents. This impossibility result, however, assumes that the entire cake must be allocated. In this paper we study the problem in a setting where the protocol may leave some of the cake un-allocated, as long as each agent obtains at least some positive value (according to its valuation). We prove that this version of the problem is solvable in a bounded time. For the case of 3 agents we provide a finite and bounded-time protocol that guarantees each agent a share with value at least 1/3, which is the most that can be guaranteed.

Categories and Subject Descriptors

F.2.2 [**ANALYSIS OF ALGORITHMS AND PROBLEM COMPLEXITY**]: Computations on discrete structures

General Terms

Algorithms, Economics

Keywords

Cake-cutting, fair division, envy-free, finite algorithm

1. INTRODUCTION

Fair cake-cutting is an active field of research with applications in mathematics, economics, and recently also in AI. The basic setting considers a heterogeneous good, usually described as a one-dimensional interval, that must be divided among several agents. The different agents may have different preferences over the possible pieces of the good, and the goal is to divide the good among the agents in a way that is deemed "fair". *Fairness* can be defined in several ways, of which *proportionality* and *envy-freeness* are the most commonly used. Proportionality means that each agent gets at least its "fair-share" of the good, i.e. with n agents, the piece allotted to each agent is worth at least $1/n$ of the value of the entire good - according to agent's subjective valuations. Envy-freeness means that no agent would prefer getting a piece allotted to another agent.

Proportional division is a relatively easy task, and a polynomial time protocol for n agents was already provided in the initial work of Steinhaus [?]. Envy-free division, on the other hand, turns out to be a much harder task. Assuming each agent needs to get a connected piece, the only protocol for envy-free division is an infinite one; that is, it may require an infinite number of queries to reach an envy-free division [?]. Indeed, Stromquist [?] proved that this is necessarily so; any algorithm for computing an envy-free division with connected pieces must require an infinite number of queries on some inputs. This is true even when there are only 3 agents!

This impossibility result seems to rule out any hope of finding a useful algorithm for computing envy-free divisions. However, a closer examination of the result reveals that it critically relies on the assumption that *the entire cake must be divided*. In many practical situations, it may be possible to leave some parts of the cake undivided, a possibility termed *free disposal*. If, for example, your children spend too much time quarrelling over the single cherry on top of the cake, one practical solution is to throw away that cherry and divide only the rest of the cake. As another example, when dividing land it is usually possible (and sometimes even preferable) to leave some parts of the land unallocated, so that they can be used freely by the public.

1.1 Results

The question of interest in this paper is thus:

> If free disposal is allowed, can an envy-free allocation be computed in bounded time?

This question, however, turns out to have a trivial, but uninteresting, answer; It is always possible to give nothing to all agents, which is an envy-free allocation. Thus, the real question is whether it is possible to devise a bounded time algorithm that computes an envy-free allocation in which each agent gets a strictly positive value. Our first result is an affirmative answer to this question.

THEOREM 1. *If free disposal is allowed, there is a bounded time protocol that for any number of agents computes an envy-free allocation giving each agent a connected piece with a positive value. The number of queries required by the protocol is only a function of the number of agents.*

Having established that bounded-time protocols indeed exists, we next consider the *quality* of the solution they offer.

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The above mentioned protocol produces an allocation that is indeed positive for all agents, but may, in worst case, give some players only a $1/2^{n-1}$ value (where n is the number of agents). An envy-free allocation of the entire cake, on the other hand, gives each agent a value of at least $1/n$. [1]

THEOREM 2. *For the case of three agents, there is a protocol with a bounded number of queries that computes an envy-free allocation giving each agent a connected piece worth at least* $1/3$, *assuming free disposal.*

This, in general, is the best possible, as there are instances in which no division can give all agents more than $1/3$.

THEOREM 3. *For the case of four agents, there is a protocol with a bounded number of queries that computes an envy-free allocation giving each agent a connected piece worth at least* $1/7$, *assuming free disposal.*

This is better than the $1/8$ bound provided by Theorem 1, but less than $1/4$ of the (non-computable) envy-free division of the entire cake. Finding better protocols for four or more agents is an interesting open question.

1.2 Related research

The cake-cutting problem comes in two variants: the harder variant requires that every agent receives a single connected piece, while the easier variant allows giving each agent a collection of disconnected pieces.

Proportional division, both for connected and disconnected pieces, is well understood from a computational perspective. The protocol of Steinhaus [**?**] requires $O(n^2)$ queries, and an improved protocol by Even and Paz [**?**] requires only $O(n \log n)$ queries. Later results proved that this runtime is asymptotically optimal, whether the pieces are connected or disconnected [**?**, **?**].

Envy-free division is a much harder task, even when only 3 agents are involved. The first envy-free division protocol for 3 agents with connected pieces was published by Stromquist [**?**]. This protocol is not discrete - it requires the agents to simultaneously hold knives over the cake and move them in a continuous manner. This means that this protocol cannot be accurately executed by a computer in finite time. A discrete and finite protocol for envy-free division for 3 agents was constructed in the same year by Selfridge and Conway [**?**], but it generates partitions with disconnected pieces.

The existence of envy-free divisions for n agents (with connected pieces) was established only by Stromquist [**?**]. This latter proof is existential in nature. The construction of a protocol for envy-free division among four or more agents was a long-standing open problem, resolved only in 1995 with the publication of the Brams-Taylor protocol [**?**]. A different protocol was later published by Robertson and Webb [**?**]. Both these protocols might generate partitions with disconnected pieces. Additionally, while these protocols are guaranteed to terminate in finite time, their runtime is not a bounded function of n. Su [**?**] presented a protocol, attributed to Forest Simmons, for envy-free division with connected pieces, but it is not finite - it converges to an envy-free division after a possibly infinite number of queries.

Stromquist [**?**] proved that an envy-free division with connected pieces cannot be found by any finite protocol, whether bounded or unbounded. This proved that the problem of connected envy-free division is more difficult than the problem of disconnected envy-free division. Shortly afterward, Procaccia [**?**] proved an $\Omega(n^2)$ lower bound on the query complexity of any envy-free division protocol, even with disconnected pieces. This proved that the problem of envy-free division is computationally more difficult than the problem of proportional division.

One way to make the envy-free division problem more manageable is to restrict the value function of the agents. Kurokawa et al [**?**] proved that if the value functions are piecewise-linear, then an envy-free division can be found in time polynomial in the size of the representation of the value functions. Their protocol might generate disconnected pieces. In contrast, our protocols always generate connected pieces, they apply to arbitrary value functions and the runtime guarantee is a function of only the number of agents, as in the classic formulation of the cake-cutting problem.

The free disposal assumption was also studied by Arzi et al [**?**]. They proved that discarding some parts of the cake may allow us to achieve an envy-free division with an improved social welfare (i.e. the sum of the utilities of the agents is larger than in the no-free-disposal case). They call this phoenomenon the *dumping paradox*. Our paper demonstrate a different kind of a dumping paradox - we show that dumping some parts of the cake can be beneficial not only from an economic perspective but also from a computational perspective. There is some related work concerning allocation of indivisible goods where the same idea of not allocating all the objects is used to get better fairness results [**?**, **?**].[2]

Partial proportionality was introduced by Edmonds and Pruhs [**?**, **?**], who used it, like us, to reduce the query complexity. They presented a protocol for finding a partially-proportional division with a query complexity of $O(n)$, which is better than the optimum of $O(n \log n)$ required for finding a fully-proportional division.

1.3 Paper structure

We proceed by formally describing our model in Section **??**. Then we present a general protocol for n agents (Section **??**) and improved protocols for 3 agents (Section **??**) and 4 agents (Section **??**).

2. THE MODEL

We assume the common 1-dimensional model in which the cake is the unit interval $[0, 1]$. The cake has to be divided among a group of n agents, giving each agent i a connected interval P_i such that the intervals given to any two different agents are disjoint.

Every agent i has a subjective value measure V_i, which is absolutely continuous with respect to length. This means that all singular points have a value of 0 to all agents, i.e. there are no valuable "atoms" which cannot be divided. The

[1] By the pigeonhole principle, the maximum value in a set is at least as large as the mean value of the set. When the entire cake is divided to n pieces, the mean value is $1/n$. In an envy-free division, each agent receives a piece whose value is (weakly) maximal, hence at least $1/n$.

[2] We thank an anonymous reviewer for referring us to these papers.

value measures are normalized such that $V_i([0,1]) = 1$.

An *envy-free partition* is a partition in which the value of an agent from his allocated interval is at least as large as his utility from every other allocated interval:

$$\forall i, j \in \{1, ..., n\} : \; V_i(P_i) \geq V_i(P_j)$$

In addition to envy-freeness, every partition can be characterized by its level of *proportionality*, which is the value of the least fortunate agent (also known as *egalitarian social welfare*):

$$Prop(\{V_i\}_{i=1}^n, \{P_i\}_{i=1}^n) = \min_{i \in \{1,..,n\}} V_i(P_i)$$

An allocation with a proportionality of $1/n$ is usually called a *proportional allocation*.

3. PROTOCOL FOR n AGENTS

Our first protocol is an adaptation of the Selfridge–Conway discrete protocol for 3 agents [**?**]. We describe it for 3 agents first. Start by imposing an arbitrary order on the three agents and calling them A, B and C.

- A cuts the cake to 3 pieces that he considers to be of equal value. Call these pieces A_1, A_2 and A_3.
- B orders the pieces according to their value according to his subjective value measure. W.l.o.g, suppose the order is: $A_3 \geq A_2 \geq A_1$.
- B cuts A_3, which is his best piece, such that there are now two pieces which he considers to be of equal value and larger than the other two. This can be done in one of two ways: (1) If $V_B(A_3) \geq 2V_B(A_2)$, then B cuts A_3 to two pieces of equal value, which is $V_B(A_3)/2$. (2) Otherwise, B cuts A_3 to two unequal pieces - one having a value of $V_B(A_2)$ and the other having a smaller value of $V_B(A_3) - V_B(A_2)$.[3]
- The agents pick their pieces in reverse order: C then B then A. Agent B is required to pick the piece that he cut, if it is available.

We now prove that the protocol generates an envy-free division with a proportionality of 1/4. The proof is based on a more general lemma, which we call the EFP (Envy-Free-Proportionality) lemma:

LEMMA 1. *(EFP Lemma) If a cake is partitioned to a set of $M \geq n$ pieces, and each agent receives a single piece that he considers to be at least as good as any other piece in that set, then the division is envy-free and its proportionality is at least $1/M$.*

PROOF. Envy-freeness is obvious since each agent receives one of his best pieces. Proportionality is a result of the fact that the value functions of the agents are measures, so they are additive. The sum of the values of all pieces is the value of the entire cake, which is normalized to 1. Hence, by the pigeonhole principle, the value of any best piece is at least $1/M$. $\square$

Going back to our cake-cutting protocol, we see that the protocol partitions the cake to $M = 4$ pieces: 3 pieces are generated by the initial division of agent A and an additional piece is generated by the cut made by agent B. Of these 4 pieces, each agent receives a piece which is at least as good as any other:

- For agent C this is obvious as he is the first to choose.
- Agent B made sure that there are 2 best pieces with equal value. When his turn arrives, at least one of those pieces is still available and he can choose it.
- Agent A made sure that there are 3 equal pieces. One of them was possibly destroyed by B and one possibly taken by C, but at least one piece is necessarily still available.

Hence, by the EFP lemma, our protocol produces an envy-free division with a proportionality of at least 1/4.

We now generalize this protocol to n agents [4]. Our main tool is the query: *Equalize*(k). When an agent is asked to *Equalize*(k), he has to cut zero or more pieces such that there are a total of k pieces which he considers to be of equal value, which is at least as good as all the other pieces.

The protocol presented above for 3 agents used two such actions: agent A was asked to *Equalize*(3), which he did by just cutting the entire cake to 3 equal pieces; agent B was asked to *Equalize*(2), which he did by cutting his best piece, either to two equal pieces or to a smaller piece which is equal to his 2nd-best piece.

For larger values of k, *Equalize*(k) becomes more complicated because there are more options. For example, for $k = 4$, the agent should either trim his 3 best pieces in a way that makes them equal to his 4th-best piece, or cut his best piece to 3 equal pieces (if each of these pieces will be at least as valuable as the 2nd-best piece), or cut his best piece to 2 equal pieces and then trim each of these pieces and his 2nd-best piece to be equal to the 3rd-best piece, etc.

Fortunately, *Equalize*(k) can be solved efficiently. In fact, it is equivalent to the following *envy-free stick division* problem: given m sticks of different lengths, make a minimal number of cuts such that there are at least k pieces with equal lengths and no other piece is longer. Reitzig and Wild [**?**] devise an algorithm that solves the envy-free stick division problem in time $O(m + \min(k, m) \log \min(k, m))$. For our purposes, it is sufficient that *Equalize*(k) can be done in bounded time.

We now return to our cake-cutting protocol. The protocol uses an integer function $P(i)$, which will be specified later. The general scheme of the protocol is as follows.

- For $i = 1$ to $n - 1$:
  - Ask agent i to *Equalize*($P(n - i)$).
- For $i = n$ to 1:

[3]If $V_B(A_3) = V_B(A_2)$, then no cutting is needed since agent B already has two pieces of equal value and better than the third piece. Here and in the rest of the paper, we ignore such fortunate coincidences because they only make the problem easier. We focus on the more difficult situation in which all pieces untouched by an agent have a different value for that agent.

[4]A reviewer has turned our attention to the fact that our generalized protocol is similar to a protocol mentioned by Brams and Taylor [**?**] (chapter 7, page 135) as a sub-routine of their unbounded protocol for envy-free cake-cutting with disconnected pieces

– Ask agent i to select one of the pieces that he trimmed, if any of them is still not taken. Otherwise, he may select any piece.

We now calculate $P(k)$. The meaning of $P(k)$ is "the number of best pieces I must have, if there are k agents cutting after me and choosing before me". We know that:

- $P(1) = 2$: Only one agent comes after me, he does not need to cut any piece and will take only one piece, so it is sufficient to have two best pieces.

- $P(2) = 3$, since the agent after me may have to destroy one piece in order to have $P(1) = 2$ equal pieces, and the last agent may take another piece (as explained above).

To calculate $P(3)$, note that the next agent may have to cut $P(2) - 1$ pieces in order to have $P(2)$ pieces that are best according to *his* measure. The first agent should have additional $P(2)$ pieces. Hence: $P(3) = [P(2) - 1] + P(2) = 2 + 3 = 5$.

We can now present a protocol for 4 agents: the first agent cuts the cake to 5 equal pieces, the second equalizes 3 pieces (by cutting at most 2 pieces) and the third equalizes 2 pieces (by cutting at most 1 piece). In total we have 8 pieces. By the EFP Lemma, by letting the agents choose pieces in reverse order, each agent receives a connected envy-free share with a value of at least 1/8.

To calculate $P(k)$, note that the next agent is going to need $P(k-1)$ best pieces, and thus may have to cut up to $P(k-1)-1$ pieces. The current agent should have $P(k-1)$ pieces which remain untouched by the next agent. Hence $P(k)$ is represented by the following recurrence relation:

$$P(k) = P(k-1) + P(k-1) - 1$$

whose solution is:

$$P(k) = 2^{k-1} + 1$$

When there are n agents, the first agent should cut the cake to $P(n-1)$ pieces. The total number of pieces is:

$$P(n-1) + \Sigma_{i=1}^{n-2}[P(i) - 1] = 2P(n-1) - 2 = 2^{n-1}$$

By the EFP Lemma, each agent receives an envy-free share with a value of at least:

$$\frac{1}{2P(n-1) - 2} = \frac{1}{2^{n-1}}$$

4. PROTOCOL FOR 3 AGENTS

The protocol of Section **??** started with an equal partition made by an arbitrary agent. In this section we achieve a better (and optimal) result for 3 agents by carefully selecting the agent which makes the initial equal partition.

Initially, each of the 3 agents is required to suggest an equal partition by marking two parallel lines that divide the cake to three subjectively equal pieces. Mark the agents: A, B and C; mark the equal pieces of agent X by: X_1, X_2 and X_3. Normalize the value functions of the agents such that the value of the entire cake is 3; hence the value of X_i to agent X is exactly 1.

Assume w.l.o.g. that the order of the first lines is A-B-C.[5] There are $3! = 6$ options for the order of the second lines, and each of these cases deserves a special treatment. The general scheme of each of these cases is as follows.

(1) Select one of the three agents (according to the case) whose initial partition will be used as the basis for the allocation. This agent will be called the "base agent" and the other agents will be called the "runners". For example, if the base agent is B then the division is based on the partition $\{B_1, B_2, B_3\}$. This means that agent B will get one of his equal pieces, and each of the runners (A and C) will get one of the other two pieces *or* a subset of it. Thus the base agent necessarily feels no envy and has a value of exactly 1. The challenge now is to make sure that the two runners also feel no envy and get a value of at least 1.

(2) Ask the two runners which of the 3 pieces they prefer. There are several cases:

Easy case: The two answers are different. Then give each runner his preferred piece and give the third piece to the base agent. Obviously there is no envy and the value per agent is at least 1, since the entire cake is divided.

If the two answers are identical, then ask the runners to evaluate their 2nd-best piece. There are two sub-cases:

Medium case: For every runner, the value of his 2nd-best piece is at least 1. Then ask each runner to $Equalize(2)$, i.e. say where the best piece should be trimmed to make it equal to his 2nd-best piece. Select the trimming in which the remaining piece is larger; say it was suggested by A. Give the trimmed piece to C and let A have his 2nd-best piece. Now both runners feel no envy and have a value of at least 1.

Hard case: For one or two runners, the value of their 2nd-best piece is less than 1. Then, a special treatment is needed to guarantee that both runners receive at least 1. We describe this special treatment in the following subsections.

For each of the 6 possible orderings of the second lines, we now specify which agent is selected as the base agent and how the division proceeds in order to guarantee that all runners receive at least 1. Recall that we assume that the order of the first lines is A-B-C, hence: $A_1 \subseteq B_1 \subseteq C_1$.

4.1 C-B-A

The base agent is **C**. Both A and B have marked no line inside C_2. This means that both runners evaluate C_2 as less than 1. Hence it cannot be their best piece; their best piece can be either C_1 or C_3. Both of the runners value both these pieces as more than 1, because $A_1 \subseteq B_1 \subseteq C_1$ and $A_3 \subseteq B_3 \subseteq C_3$. Hence, both runners have a 2nd-best piece with a value of more than 1, and the hard case never happens.

4.2 C-A-B

The analysis of the case C-B-A applies as is to this case.

4.3 A-B-C

The base agent is **B**. $A_3 \supseteq B_3 \supseteq C_3$, hence A prefers either B_1 or B_2 and agent C prefers either B_2 or B_3. Hence, if both of them prefer the same piece, it must be B_2. In this case, the 2nd-best piece of A is B_1 which contains A_1 so A values it more than 1; similarly, the 2nd-best piece of C if

[5]Again we ignore the case in which two or more agents make a mark in the exact same spot. This case can be handled by assuming an arbitrary order between these agents.

B_3 which contains C_3 so C values it more than 1. Hence again the hard case never happens.

4.4 B-A-C

The base agent is **B**. $B_3 \supseteq A_3 \supseteq C_3$ and $B_2 \subseteq A_2$, hence agent A prefers either B_1 or B_3 and agent C prefers either B_2 or B_3. Hence, if both of them prefer the same piece, it must be B_3. The 2nd-best piece for agent A is B_1 which containsA_1, so its value for A is more than 1. However, for agent C it is possible that its 2nd-best piece, B_2, has a value of less than 1 (the hard case). In this case, allocate each agent one of his equal pieces (having a value of exactly 1), in the following way:

- A_1 to agent A. By the containment $A_1 \subseteq B_1 \subseteq C_1$, its value for the other agents is less than 1 so they feel no envy.
- C_3 to agent C. By the containment $B_3 \supseteq A_3 \supseteq C_3$, its value for the other agents is less than 1 so they feel no envy.
- B_2 to agent B. By the containment $B_2 \subseteq A_2$, A values this piece as less than 1; by the assumption of the hard case, C also values this piece as less than 1, so both feel no envy.

4.5 A-C-B

The previous case, A-B-C-B-A-C, is symmetric to A-B-C-A-C-B. This can be seen by renaming the agents from A-B-C to B-C-A and reversing the order of lines.

4.6 B-C-A

This last sub-case is the most complicated. First, ask agent A which of the two pieces he prefers: B_1 (which contains A_1) or C_3 (which contains A_3). Note that A values both these pieces as more than 1. Proceed according to the answer:

If agent A prefers B_1, then find a division based on **B**'s partition, similarly to the case B-A-C. The only change required is in the handling of the hard case. In this case, make the following allocation:

- B_1 to agent A. By the containment $A_1 \subseteq B_1 \subseteq C_1$, its value for the other agents is at most 1 so they feel no envy.
- C_3 to agent C. By the containment $B_3 \supseteq C_3$, its value for B is less than 1 so B feels no envy; by A's initial choice, A also feels no envy.
- B_2 to agent B. By the containment $B_2 \subseteq A_2$, A values this piece as less than 1; by the assumption of the hard case, C also values this piece as less than 1, so both feel no envy.

If agent A prefers C_3, then find a division based on **C**'s partition, using a symmetric protocol. In the hard case, make the following allocation:

- C_3 to agent A; by the containment $B_3 \supseteq C_3 \supseteq A_3$, its value for the other agents is at most 1 so they feel no envy.
- B_1 to agent B; by the containment $B_1 \subseteq C_1$, C feels no envy; by A's initial choice, A also feels no envy.
- C_2 to agent C. By the containment $C_2 \subseteq A_2$, A values this piece as less than 1; by the assumption of the hard case, B also values this piece as less than 1, so both feel no envy.

5. PROTOCOL FOR 4 AGENTS

Encouraged by the performance of the protocol of Section **??**, we would like to extend it to produce an envy-free and proportional allocation for n agents. Unfortunately, the number of different cases becomes prohibitively large even for $n = 4$ agents. The equal partition of each agent is made by 3 parallel marks, so if we name the agents according to their 1st mark, the number of options for the following two marks is $(4!)^2 = 576$, and in general $(n!)^{n-2}$. The protocol for each specific case may be short, but writing down all the different cases takes too long to be practical.

This section presents a different technique and uses it to develop an envy-free allocation protocol for 4 agents with a proportionality of 1/7, which is better than the 1/8 guaranteed by the protocol of Section **??**. We believe that this technique may be used for achieving better results in future work.

The technique involves the *preference graph* - a bi-partite graph in which the nodes in one partition represent the n agents and nodes in the other partition represent the currently available (m) pieces of the cake. There is an edge from an agent X to a piece i if agent X *prefers* piece i, i.e., $\forall j \in \{1, ..., m\} : V_X(i) \geq V_X(j)$. Note that an agent can "prefer" two or more pieces. This means that the agent is indifferent between these pieces but values any of them more than any other piece. Here are two possible preference graphs for 3 agents:

Both graphs may be the result of agent A cutting the cake to 3 equal pieces. In the left graph, B and C each prefer a different piece; in the right graph, they prefer the same piece (3).

A *matching* in the preference graph represents an allocation of pieces to agents. We call a matching *saturated* if all agent nodes are matched (note that we do not require the matching to be *perfect* since we do not require that all piece nodes be matched).

By the EFP lemma, if a matching covers all n agents then the corresponding allocation is envy-free and has a proportionality of at least $1/m$. So the problem of finding an envy-free allocation reduces to finding a saturated matching in a preference graph.

A well-known tool for proving the existence of saturated matchings in bi-partite graphs is Hall's marriage theorem. This theorem, applied to our setting, implies that *an envy-free division exists iff every group of k agents joinly prefers at least k pieces.* In the left graph above, Hall's condition is satisfied, which means means that there is an envy-free division with a proportionality of 1/3.

In the right graph above, Hall's condition is violated by the group {B,C}. This means that an envy-free division using the existing pieces is impossible. In this case, the graph should be *transformed* in order to create a graph that meets Hall's condition. We apply transformations based on the *Equalize* action.

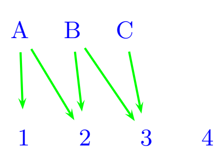
We use a variant of *Equalize* which simultaneously asks several agents to suggest an equalizing cut of a given piece. For example, a possible action is: "ask agents {B,C} to $Equalize(2, \widehat{i})$". The first argument, 2, is the number of equal-value pieces resulting from the action. The second input, $\widehat{i}$, is a certain piece of the cake - a certain node in the graph (for clarity we write piece numbers below a hat). Such a query makes sense only if B and C currently prefer piece $\widehat{i}$. The query requires an agent to indicate where piece $\widehat{i}$ should be cut so that the agent will prefer 2 pieces. The agent has to suggest either a *trimming* that will make $\widehat{i}$ equal to his 2nd best piece, or a *halving* that will divide $\widehat{i}$ into two equal-value pieces (in case the current value of $\widehat{i}$ is more than twice the value of the 2nd-best piece).

The protocol always implements the *mildest cut* - the cut which leaves the largest reminder. Suppose the mildest cutter is agent X. The effect of the action on the graph is as follows:

- A new piece node is added (i.e. m grows by 1).
- A new edge is added from agent X to another piece. In case of a trimming, the new edge is to an existing piece which previously was X's 2nd-best piece. In case of a halving, the new edge is to the new piece.
- All edges from other agents to piece $\widehat{i}$ are removed, since the piece has now changed.
- For every agent Y that has no outgoing edges, a new edge is added to Y's new best piece. If Y is in the group of agents that were asked to Equalize (the group {B,C} in our example), then this new edge must be to piece $\widehat{i}$. This is because the mildest cut was implemented, so the remaining piece $\widehat{i}$ contains a piece which is equal to their 2nd-best piece.

Going back to the right preference graph above, in which both B and C prefer piece 3, we now ask {B,C} to *Equalize* $(2, \widehat{3})$. This action has the following outcomes: (*) It adds a new piece $\widehat{4}$. (*) It creates an edge from the mildest cutter (which can be either B or C; w.l.o.g. we assume it is B) to his 2nd-best piece (which again w.l.o.g. we assume to be $\widehat{2}$). (*) It removes the edge A-$\widehat{3}$. The edge C-$\widehat{3}$ is kept because C is in the group that was asked to Equalize.

Now Hall's condition is met and we have an envy-free division with a proportionality of 1/4:



In order to reduce the number of cases to handle, we make two assumptions:

(a) We assume that B and C have only a single outgoing edge. In general, we assume that an agent can prefer two pieces (i.e. assign the same maximal value to two pieces), only if that agent made specific cuts guaranteeing that these pieces have the same value. So when agent A cuts the cake to 4 equal pieces, we assume that every other agent assigns different values to the resulting pieces and thus prefers only a single piece. This assumption does not lose generality, because it only decreases the number of edges, and thus makes it more difficult to find a saturated matching. In other words, if agent B happens to prefer more than one piece from A's cut, we arbitrarily remove all but a single edge, since every saturated matching in the reduced graph is also a saturated matching in the original graph.

(b) We assume that the new piece 4 is not liked by anyone. This assumption is also justified because, from Hall's perspective, it only makes our task more difficult.

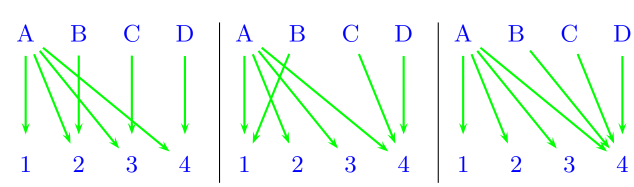
In the following analysis we always make these assumptions and also omit the new pieces in the graphs, keeping in mind the total number of pieces for the proportionality calculations.

In the protocol for 4 agents we use both $Equalize(2, \widehat{i})$ and $Equalize(3, \widehat{i})$. The latter query can be sent to agents for whom piece $\widehat{i}$ is currently the best or the 2nd-best piece. It has the following meaning: each agent is asked where piece $\widehat{i}$ (and one additional piece) should be cut so that the agent will prefer 3 pieces.

The protocol selects the mildest cut - the cut that leaves the largest remainder of piece $\widehat{i}$. Suppose that the mildest cutter of piece $\widehat{i}$ is X and that X chose to also cut piece $\widehat{j}$ (which was his 2nd-best piece). The protocol cuts both $\widehat{i}$ and $\widehat{j}$ as suggested by X. The effect on the graph is as follows:

- Two new piece nodes are added.
- Two new edges are added from agent X to other pieces. Each edge can be either to an existing piece (which previously was X's 2nd-best piece $\widehat{j}$ or X's 3rd-best piece) or to a new piece.
- All edges from other agents to pieces $\widehat{i}$ and $\widehat{j}$ are removed, since these pieces are smaller now.
- For every agent Y that has no outgoing edges, a new edge is added to Y's new best piece. If Y is in the group of agents asked to Equalize, we can be sure that the piece $\widehat{i}$ is now better than Y's 3rd-best piece (since the mildest cut of piece $\widehat{i}$ was selected). So there are two possibilities: (a) Piece $\widehat{i}$ is Y's best piece; (b) Piece $\widehat{i}$ is Y's 2nd best piece; in that case, another piece, which was previously Y's 2nd-best piece, is now Y's best piece.

The division protocol begins with an arbitrary agent (A) cutting the cake to 4 equal pieces. We proceed according to the number of neighbours of the agents {B,C,D}. Recall that we assume that each of these agents has a single neighbour. Hence there are three cases: either they have in common 3 neighbours (left), 2 neighbours (middle) or 1 neighbour (right):



5.1 3 neighbours

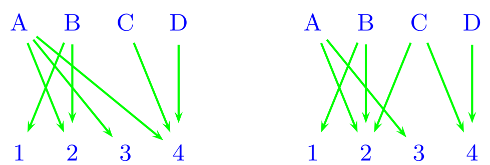
Hall's condition is met with 4 pieces. Therefore there is a saturated matching which represents an envy-free division with proportionality 1/4.

5.2 2 neighbours

Hall's condition is violated for {C,D}. We would like to correct this by asking {C,D} to $Equalize(2, \widehat{4})$, but this may create a conflict with B, so some preparation is needed.

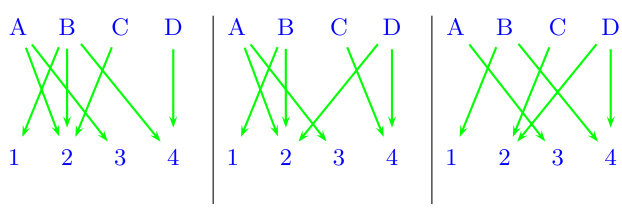
Begin by checking what is the 2nd-best piece of B. By "2nd best" we mean the 2nd piece that will be preferred by B if B does $Equalize(2, \widehat{1})$. This can be either an existing piece ($\widehat{2}$, $\widehat{3}$ or $\widehat{4}$, in case B decides to trim $\widehat{1}$) or a new piece ($\widehat{5}$, in case B decides to half $\widehat{1}$). We proceed according to the following cases:

Easy case: the 2nd-best piece of B is $\widehat{2}$ or $\widehat{3}$ or $\widehat{5}$ (i.e., different than the best piece of C and D). Ask B to $Equalize(2, \widehat{1})$ and get a graph like the one at the left (we assumed w.l.o.g. that B's 2nd-best piece is $\widehat{2}$; note the edge A-$\widehat{1}$ was removed and an edge B-$\widehat{2}$ was added). Next, ask {C,D} to $Equalize(2, \widehat{4})$ and get a graph like the one at the right (we assumed w.l.o.g. that the mildest cutter was C; we also assumed that his 2nd-best piece was also $\widehat{2}$, which is the worst case). Now Hall's condition is met with 6 pieces:

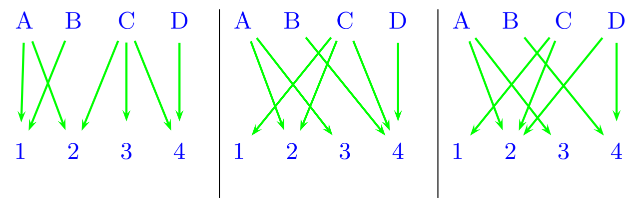


Hard case: the 2nd-best piece of B is $\widehat{4}$. This means that for all three agents B, C and D, piece $\widehat{4}$ is more valuable than their 3rd-best piece, so we can ask {B,C,D} to $Equalize(3\widehat{4})$. There are now two sub-cases.

– *Subcase 1*: the mildest cutter of $\widehat{4}$ is B, so there are edges from B to $\widehat{1}$ and $\widehat{4}$ and another piece (say, $\widehat{2}$). We also know that for C and D, $\widehat{4}$ is now either their best or their 2nd-best piece, since it is better than their 3rd-best piece. If exactly one of {C,D} prefers $\widehat{4}$, then Hall's condition is met with the existing 6 pieces (left). If both of C and D prefer $\widehat{4}$, then ask them to $Equalize(2, \widehat{4})$ and Hall's condition is satisfied (middle); If both C and D prefer another piece (say, $\widehat{2}$), then ask them to $Equalize(2, \widehat{2})$. This will make one of them prefer $\widehat{4}$ and again Hall's condition will be satisfied (right; both graphs illustrate the case that the mildest cutter is D):



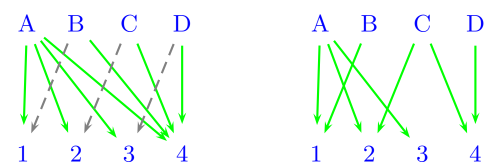
– *Subcase 2*: the mildest cutter of $\widehat{4}$ is C (or equivalently D). This means that piece $\widehat{4}$ and one additional piece were trimmed by C. If that additional piece is $\widehat{2}$, $\widehat{3}$ or a new piece, then the new graph satisfies Hall's condition regardless of which piece was C's 3rd-best (left, assuming the additional trimmed piece is $\widehat{3}$). The harder case is that C's 2nd-best piece is $\widehat{1}$, and C trims it so much that it is no longer preferred by B (middle). So now B prefers piece $\widehat{4}$ and Hall's condition is violated by {B,D}. Ask {B,D} to $Equalize(2, \widehat{4})$ and the graph will satisfy Hall's condition with 7 pieces (right):



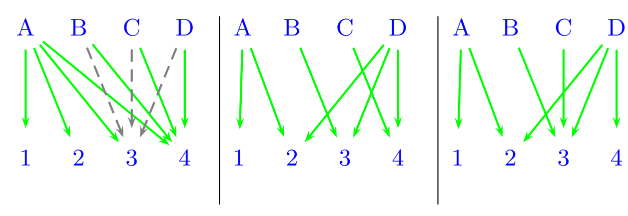
5.3 1 neighbour

This means that B, C and D all prefer the same piece (say, $\widehat{4}$). There are three cases.

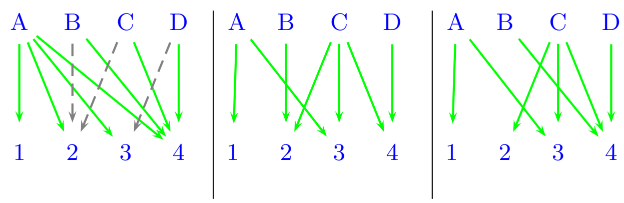
Easy case: each player has a different 2nd-best piece, say, the 2nd-best piece of B is $\widehat{1}$, of C is $\widehat{2}$ and of D is $\widehat{3}$ (left; dashed line indicates 2nd-best piece). Send two *Equalize* queries on $\widehat{4}$, e.g. ask {B,C,D} to $Equalize(2, \widehat{4})$ and then (assuming the mildest cutter was B) ask {C,D} to $Equalize(2, \widehat{4})$ again. This leads to a graph similar to the one at the right (assuming the mildest cutter in the second trimming was C), which satisfies Hall's condition with 6 pieces:



Medium case: all players have the same 2nd-best piece, say, $\widehat{3}$ (left). The case in which the 2nd-best piece is one of the new pieces, i.e. $\widehat{5}$ or $\widehat{6}$, is similar. Ask {B,C,D} to $Equalize(3, \widehat{4})$. Suppose w.l.o.g. that the mildest trimmer is D and that his 3rd best piece is $\widehat{2}$. For each agent in {B,C} there are two possibilities: either his best piece is still $\widehat{4}$, or his best piece is $\widehat{3}$ and his 2nd-best piece is $\widehat{4}$. If the best pieces are different, then Hall's condition is satisfied with the existing 6 pieces (middle). If the new best piece of {B,C} is the same, say, $\widehat{3}$ (right), then ask {B,C} to $Equalize(2, \widehat{3})$. This will make one of them prefer $\widehat{4}$ and satisfy Hall's condition with 7 pieces:



Hard case: two players have the same 2nd-best piece, say, the 2nd-best piece of B and C is $\widehat{2}$ and of D is $\widehat{3}$ (left). Ask {B,C,D} to $Equalize(3, \widehat{4})$. If the mildest cutter is D then the situation is identical to the medium case. If the mildest cutter is C (or equivalently B), then the situation is similar, since D prefers either $\widehat{4}$ or $\widehat{3}$ and B prefers either $\widehat{4}$ or $\widehat{2}$. If their best pieces are different, then Hall's condition is satisfied with the existing 6 pieces (middle); if both of them prefer $\widehat{4}$ (right), then ask {B,D} to $Equalize(2, \widehat{4})$ and Hall's condition will be satisfied with 7 pieces:



To summarize this section, we have shown that it is possible to achieve a graph satisfying Hall's condition with at most 7 pieces. This means that it is possible to have an envy-free division to 4 agents with a proportionality of at least 1/7.

6. CONCLUSION AND FUTURE WORK

We proved that the problem of envy-free division with connected pieces can be solved in finite, bounded time if we allow to leave some parts of the cake unallocated. For the case of 3 agents, this does not require a reduction in the guaranteed minimal value per agent, since it is possible to guarantee that each agent receieves at least his fair share of 1/3 the total value.

A challenging task for future work is to improve the proportionality bounds for $n \geq 4$ agents. The protocol of Section **??**, which uses a small number of actions with a finite number of possible outcomes for each action, suggests that it may be possible to utilize AI planning tools for constructing division protocols when the number of agents is sufficiently small.

Our protocols assume that each agent must receive a single connected piece. If this requirement is relaxed and each agent may get several disconnected pieces, it may be possible to attain better proportionality bounds.

It is an interesting open question whether an envy-free and proportional division is attainable in bounded time for 4 or more agents.

7. ACKNOWLEDGEMENTS

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