Computing Optimal Coalition Structures in Polynomial Time*

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ABSTRACT

The optimal coalition structure determination problem is in general computationally hard. In this article, we identify some problem instances for which the space of possible coalition structures has a certain form and constructively prove that the problem is polynomial time solvable. Specifically, we consider games with an ordering over the players and introduce a *distance metric* for measuring the distance between any two structures. In terms of this metric, we define the property of *monotonicity*, meaning that coalition structures closer to the optimal, as measured by the metric, have higher value than those further away. Similarly, *quasi-monotonicity* means that part of the space of coalition structures is monotonic, while part of it is non-monotonic. (Quasi)-monotonicity is a property that can be satisfied by coalition games in characteristic function form and also those in partition function form. For a setting with a monotonic value function and a known player ordering, we prove that the optimal coalition structure determination problem is polynomial time solvable and devise such an algorithm using a greedy approach. We extend this algorithm to quasi-monotonic value functions and demonstrate how its time complexity improves from exponential to polynomial as the degree of monotonicity of the value function increases. We go further and consider a setting in which the value function is monotonic and an ordering over the players is known to exist but ordering itself is unknown. For this setting too, we prove that the coalition structure determination problem is polynomial time solvable and devise such an algorithm.

KEYWORDS

Coalition formation; Characteristic function games; Partition function games

ACM Reference Format:
Shaheen Fatima and Michael Wooldridge. 2019. Computing Optimal Coalition Structures in Polynomial Time. In *Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), Montreal, Canada, May 13–17, 2019,* IFAAMAS, 2 pages.

1 INTRODUCTION

We address the problem of determining an optimal coalition structure, i.e., a partition of a set of n plasyers into disjoint coalitions so as to optimize the value of the partition. It is assumed that players are ordered by their priorities; the higher a player's priority, the more important it is to place it correctly. We introduce a *distance metric* for measuring how close any two partitions are. In terms of this metric, we define a property of value functions that we call *monotonicity*. Intuitively, monotonicity means that coalition structures that are closer to the optimum (when measured using the distance metric) have higher value than those further away.

For games with a known player ordering we show how to determine optimal coalition structure in polynomial time. Then we consider games in which player ordering is unknown and show how the optimal coalition structure can still be determined in polynomial time.

Placing our contribution in the context of existing literature, we note that existing research has been devoted to the coalition structure generation problem, but most of it has focused on the complete set partitioning problem, with even the best solutions having exponential time complexity. As for PFGs, the search space is considerably larger relative to CFGs. Some recent literature has dealt with restricted cases of PFGs: in [4] either positive only or negative only externalities are allowed but not both, while in [1] mixed externalities are considered but only for one specific value function. In other research [6], the computational complexity is overcome by imposing restrictions on the size of coalitions that can form. Apart from these, some heuristic methods have also been studied [5].

The key distinctive features of the proposed methods in this paper are:
i) Unlike existing methods, they are suitable for any kind of value function (i.e., non-separable, CFGs, and PFGs with positive only, negative only, and mixed externalities) that satisfies monotonicity.
ii) Unlike existing methods, they require only an ordering on the values of partitions to be known but not their actual values. The methods in literature require the actual value of each coalition to be known, and assume that the value function is separable in that the value of each partition is simply the sum of the values of the coalitions in it. The proposed methods are therefore practically more relevant because it is easier to know an ordering over the values of partitions but much harder to know their exact values, especially for large games.

2 THE MODEL

We assume a finite, non-empty set of *players* $N = \{1, \ldots, n\}$. A *coalition*, C, is simply a subset of N, i.e., $C \subseteq N$. We will denote the set $\{1, \ldots, i\}$ by $[i]$.

A central concept in our work is the notion of a *coalition structure*. Intuitively, a coalition structure captures the idea of the players N dividing into separate coalitions, and conventionally, a coalition structure is therefore simply defined as a partition of N [2]. Without any loss of generality, assume that the coalitions in a structure π are ordered by the following principle:

*This paper is an extended abstract of an article published in Autonomous Agents and Multi-Agent System [3].

Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), N. Agmon, M. E. Taylor, E. Elkind, M. Veloso (eds.), May 13–17, 2019, Montreal, Canada.

coalition C_i will precede a coalition C_j in π if the smallest element of C_i is less than the smallest element of C_j.

Note that since coalitions in a coalition structure are assumed to be mutually disjoint, then this ordering is guaranteed to be strict. Formally, we have:

Definition 2.1. A coalition structure $\pi = (C_1, C_2, \ldots, C_M)$ over N is a sequence of coalitions $C_i \subseteq N$ such that:

(1) $\bigcup_{1 \le i \le M} C_i = N$,
(2) $C_i \cap C_j = \emptyset$ for $1 \le i \le M$, $1 \le j \le M$ s.t. $i \ne j$, and
(3) for $1 \le i \le M$ and $1 \le j \le M$, if $i < j$ then $\min C_i < \min C_j$.

with the convention $\min \emptyset = \infty$.

Let Π_N denote the set of all coalition structures over N. We will use the terms coalition structure, sequence, and partition synonymously. Observe that, defined in this way, coalition structures have the following property. In any coalition structure, player 1 must belong to the first coalition, player 2 must belong to one of the first two coalitions, and so on. In general, if the players $1, \ldots, i$ ($1 \le i < n$) belong to the first $1 \le m \le i$ non-empty coalitions of any sequence, then player $i+1$ must belong to one of the first $m+1$ coalitions in it.

We find it useful to work with a functional representation of coalition structures, which we call the *sequence form*. A sequence form representation is simply a function that maps a given player i to the index of the coalition of which i is a member. More generally, it is useful to work with sequence form functions that only define coalition membership for some subset of the overall set of agents:

Definition 2.2. For $1 \le k \le n$, a **sequence form**

$$\mathrm{SF}_k : [k] \to N$$

maps each player $i \in [k]$ to the index of a coalition in a sequence. An *instance of SF_k* is any coalition structure π of N that satisfies the condition

$$\forall i \quad i \in [k] \Rightarrow \text{player } i \text{ belongs to the coalition } \mathrm{SF}_k(i) \text{ in } \pi. \square$$

Each coalition structure has a *value* given by a function v:

$$v : \quad \Pi_N \to \mathbb{R}$$

A pair (N, v) constitutes a *coalition game*. Given a coalition game, the problem is to find an **optimal sequence** π^{OPT} such that:

$$\pi^{\mathrm{OPT}} \quad \in \quad \underset{\pi \in \Pi_N}{arg\ max} \quad v(\pi).$$

This problem is, in general, computationally hard because of the huge search space. For a game of n players, the number of all possible coalition structures is given by $Bell(n)$. Since $Bell(n) \sim \Theta(n^n)$, it is, in general, infeasible to compute an optimal coalition structure by exhaustive search over the space of all coalition structures.

The aim is to identify value functions that are practically relevant and for which the coalition structure generation problem can be solved in polynomial time. We consider value functions that are *quasi-monotonic*. Monotonicity is defined in terms of a *distance metric* on the space Π_N of possible coalition structures.

The metric we consider is defined in terms of the notion of *restriction* of a partition to a coalition.

Definition 2.3. The **restriction** $\pi_{|[i]}$ of a coalition structure $\pi = (C_1, C_2, \ldots)$ to coalition $[i]$ is defined as

$$\pi_{|[i]} = (C_1 \cap [i], C_2 \cap [i], \ldots).$$

For readability, any empty sets will not be shown in $\pi_{|[i]}$.

For any two coalition structures over N, we have the following readily established property.

Lemma 2.4. *Let $\pi^1 \in \Pi_N$ and $\pi^2 \in \Pi_N$ be any two coalition structures over $N = \{1, \ldots, n\}$. Then $\exists u \in \mathbb{N}$ such that $\pi^1_{|[u]} = \pi^2_{|[u]}$.*

Lemma 2.4 can be proved easily by letting $u = 1$. Intuitively, the *larger* the value of u, then the “closer” π^1 and π^2 are together. This motivates the introduction of the following *distance metric d*, which gives a measure of the distance between any two elements in Π_N:

$$d(\pi^1, \pi^2) \quad = \quad \frac{1}{\Delta(\pi^1, \pi^2)} \tag{1}$$

where $\Delta(\pi^1, \pi^2) = \max\{i \in \mathbb{N} : \pi^1_{|[i]} = \pi^2_{|[i]}\}$.

For the case of unique optimum, we assume monotonicity: the function $v : \Pi_N \to \mathbb{R}$ is monotonically decreasing in the distance of a sequence from the optimum π^{OPT}, i.e., for two arbitrary sequences π^1 and $\pi^2 \ne \pi^1$, we have the following implication:

$$d(\pi^{\mathrm{OPT}}, \pi^1) < d(\pi^{\mathrm{OPT}}, \pi^2) \quad \Rightarrow \quad v(\pi^1) > v(\pi^2) \tag{2}$$

Lemma 2.5. *For $1 \le i < n$, any sequence in which the first mis-placed player is $i+1$ (i.e., the players $1, \ldots, i$ are correctly placed in their respective optimal coalitions but not player $i+1$) has a higher value than any sequence in which the first mis-placed player is i (i.e., the players $1, \ldots, i-1$ are correctly placed in their respective optimal coalitions but not player i).*

Lemma 2.5 leads to the definition of player priority.

Definition 2.6. Each player has a priority: player 1 has the highest priority and the priority of any $1 \le i < n$ is higher than that of player $i+1$, i.e., $1 \succ 2, \ldots, \succ n$.

We consider games with known player priorities and show how to determine optimal coalition structure in polynomial time. Then we consider games in which the priorities are unknown and show how the optimal coalition structure can still be determined in polynomial time.

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