Testing Individual-Based Stability Properties in Graphical Hedonic Games

Hendrik Fichtenberger*
TU Dortmund, Germany
hendrik.fichtenberger@tu-dortmund.de

Amer Krivošija
TU Dortmund, Germany
amer.krivosija@tu-dortmund.de

Anja Rey
TU Dortmund, Germany
anja.rey@tu-dortmund.de

ABSTRACT

In hedonic games, players form coalitions based on individual preferences over the group of players they belong to. Several concepts to describe the stability of coalition structures in a game have been proposed and analysed. However, prior research focuses on algorithms with time complexity that is at least linear in the input size. In the light of very large games that arise from, e.g., social networks and advertising, we initiate the study of sublinear time property testing algorithms for existence and verification problems under several notions of coalition stability in a model of hedonic games represented by graphs with bounded degree. In graph property testing, one shall decide whether a given input has a property (e.g., a game admits a stable coalition structure) or is far from it, i.e., one has to modify at least an ϵ-fraction of the input (e.g., the game's preferences) to make it have the property. In particular, we consider verification of perfection, individual rationality, Nash stability, and (contractual) individual stability. Furthermore, we show that while there is always a Nash-stable coalition (which also implies individually stable coalitions), the existence of a perfect coalition can be tested. All our testers have one-sided error and time complexity that is independent of the input size.

KEYWORDS

cooperative games; sublinear algorithms; hedonic games; stability; property testing

ACM Reference Format:
Hendrik Fichtenberger, Amer Krivošija, and Anja Rey. 2019. Testing Individual-Based Stability Properties in Graphical Hedonic Games. In *Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), Montreal, Canada, May 13–17, 2019,* IFAAMAS, 9 pages.

1 INTRODUCTION

Hedonic games are a form of coalition formation games, in which players form teams in a decentralized manner based on individual preferences over coalitions, i.e., subsets of players. The solution to such a game is a coalition structure, i.e., a partition of the set of players. The main idea of hedonic games is that the players' evaluation of a coalition structure only depends on their own coalitions and not on how other players work together [18]. These games have been formalized by Banerjee et al. [6] and Bogomolnaia and Jackson [9], independently. In order to evaluate the quality of a coalition structure, several solution concepts have been considered. These include, e.g., Nash stability, which states that no individual player wants to deviate from the coalition structure, and core stability, which requires that no group of players wants to peel off and form a coalition of their own.

Algorithmically, a key issue is to find suitable representations of hedonic games: Since the number of possible coalitions for a player is exponential in the number of players, there is a trade-off between compactness and expressivity of the preference profile. In the areas of *Cooperative Games in Multiagent Systems* (see, e.g., [12]) and *Computational Social Choice* (see, e.g., [4]), a number of representations and stability concepts are analysed with respect to the computational complexity of deciding whether there exists a stable solution, verifying whether a given solution is stable, and finding a stable solution. Even for restricted representations such as additively separable games, these questions are often intractable. For instance, it is often NP-complete to decide whether a given game allows a Nash-stable coalition structure, see e.g., Peters [34]. The existence of core-stability is often even Σ_2^p-complete to decide, see e.g., Woeginger [38] and Ota et al. [33]. This strikes even harder when the considered game instances are very large because they arise from, e.g., social networks or the assignment of advertisements to available slots on web pages such that adjacent ads do not interfere. Here, it might already be impractical to read the whole input once because the data does not fit into memory or the access is slow or restricted.

In this paper, we study sublinear algorithms for hedonic games. We aim to decide in sublinear time whether a game has a stable coalition structure or is far from this with respect to the number of required changes of preferences such that it admits one, as well as whether a game is stable under a given coalition structure Γ or is far from being stable under Γ. When a coalition can be stabilized by only few compromises on the preferences, it may be acceptable to sustain the situation or (if possible) make the changes. When, however, too many modifications are required to obtain any stable situation, the current situation is too far off the goal.

Graph representations provide a compact means to encode structural connections between players. A formal study of graphical hedonic games is provided by Peters [34]. A popular variant is to encode a game as a network where players correspond to vertices and edges illustrate friendship relations. Players that are no friends are often referred to as enemies. Preferences are extended either by prioritising appreciation of friends or aversion to enemies [17]. However, if the game is very large, many players may not be involved in any relationship. In this scenario, it is natural to consider a more general model. For each player, the set of other players is divided into three subsets: friends, enemies and neutral players [33],

*This author acknowledges the support by ERC grant No. 307696.

Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), N. Agmon, M. E. Taylor, E. Elkind, M. Veloso (eds.), May 13–17, 2019, Montreal, Canada.

which is what we call the FEN-encoding. Furthermore, we bound the number of friends and enemies per player by a constant (for example, if the players are humans, this phenomenon is known as *Dunbar's number* [19], which describes the maximum number of stable relationships a single person can maintain). Under restrictions such as bounded degree and bounded treewidth, some stability questions become solvable in linear time [34]. Nevertheless, this still incurs the evaluation of the whole game in order to verify whether a coalition structure is stable. Given the local views of individual vertices within hedonic games, it would be preferable and much more practical to ask only a sample of players for their individual preferences and deduce global properties.

The area of property testing provides a framework to relax such decision problems in favour of sublinear complexity (see, e.g., [23] for an overview). A property tester is a randomized algorithm that decides, with error probability at most 1/3, whether the input satisfies some property $\mathcal{P}$ or is far from satisfying $\mathcal{P}$ by probing only a small part of it. In the setting of graph properties, a graph $G = (V, E)$ with bounded vertex degree d is ϵ-far from satisfying some property $\mathcal{P}$ (e.g., bipartiteness) if one has to modify at least ϵdn edges to make G have property $\mathcal{P}$. If the property tester always accepts graphs in $\mathcal{P}$, it has one-sided error; otherwise, it has two-sided error. The input graph G may be probed by the algorithm through an oracle that provides access to the entries of the adjacency lists of G, and the computational complexity of the property tester is measured in terms of queries it asks.

In comparison to classic decision problems, property testing problems allow for algorithms with sublinear complexity. For example, a randomized decision algorithm for graph connectivity needs to read the whole input to achieve constant error probability, which implies a linear lower bound on the complexity. In contrast, a property tester for connectivity has only constant complexity [22]. This difference arises because the property tester does not need to read the whole input, and, in fact, sublinear complexity renders this impossible. Therefore, the input model plays an important role in property testing. While there is a characterization for constant query testable properties in dense graphs (graphs with $\Omega(|V|^2)$ edges) [1], less is known for graphs with bounded degree and general graphs.

1.1 Our Contribution

We study property testing of stability problems in FEN-hedonic games, where each player has a bounded number of symmetric relationships to friends and enemies as represented by labelled edges of an undirected graph, and preferences are extended to coalitions by any utility function linear in the number of friends and enemies in a coalition. The setting of hedonic games enhances graphs by rich semantics, which stands in contrast to purely combinatorial and geometric properties previously studied in graph property testing. We model the semantics of hedonic games as an additional layer on top of the combinatorial graph structure and analyse existence and verification problems for various stability concepts. In particular, we study common individual-based stability concepts such as perfection, individual rationality, Nash stability, and (contractual) individual stability.

While individually rational, Nash-stable, individually stable, and contractually individually stable coalitions always exist, there are games which do not allow a perfect coalition structure.

Theorem 1.1. *Given a FEN-hedonic game G with bounded degree d, it can be tested whether G admits a perfect coalition structure with bounded coalition size c with one-sided error and query complexity* $\mathrm{poly}(\epsilon, c, d)$.

While the existence problem as to whether a game allows a stable outcome is a property of edge-labelled graphs, the verification problem of whether a game satisfies stability according to a fixed coalition structure Γ requires additional modelling: We assume that next to oracle access to the adjacency lists of the underlying bounded-degree graph of a game G, we have additional access to an oracle to Γ, i.e., a partition of the vertex set.

We show the testability of verification problems independent of any bound on the coalition size.

Theorem 1.2. *Given a FEN-hedonic game G with bounded degree d and a coalition structure Γ, it can be tested whether G is stable under Γ with respect to perfection, individual rationality, Nash stability, individual stability and contractual individual stability with one-sided error and query complexity* $\mathrm{poly}(\epsilon, d)$.

Note that while we consider c and d to be of constant size, independent of the input size n, our statements remain valid if, for instance, $d \in O(\log n)$. We provide some extensions of our theorems to weighted and directed graphs in Section 4.3.

1.2 Related Work

Hedonic games were formally defined by Banerjee et al. [6] and Bogomolnaia and Jackson [9]. A well-known application of a restricted variant with size-two coalitions is the stable-roommates problem [10] for the allocation of student houses. Mostly, hedonic games have been analysed from a computational complexity point of view with respect to a trade-off between expressivity, succinct representation and tractability of stability decision problems. The complexity of general hedonic games has first been studied by Ballester [5]. The worst-case complexity of stability problems for various representations and different stability concepts has been studied extensively: Popular representations include additively separable hedonic games [2, 9, 38], singleton encodings [11], hedonic coalition nets [21], and dichotomous preferences [3]; see also Aziz and Savani [4] and Chalkiadakis et al. [12] for an overview. Peters and Elkind [35] analyse causes of and conditions for hardness. The existence of Nash stability and other individual stability concepts is often (if not guaranteed) NP-complete to decide (see, e.g., Sung and Dimitrov [37] for additively separable games). For core stability, this is often even harder, namely Σ_2^p-complete [33, 38]. Dimitrov et al. [17] define restricted hedonic games based on a network of friends and enemies. A more general version including neutral players is defined by Ota et al. [33]. Games with neutral players and partial individual evaluations are studied by Lang et al. [28] and Peters [34]. Peters, in particular, considers a constant bound on the number individual preferences and studies graphical hedonic games with bounded treewidth. With this restriction, it can be decided in linear time whether, for instance, a Nash stability coalition structure exists. A graphical model restricting the formation of feasible coalitions is studied by Igarashi and Elkind [26]. Darmann et al. [16] study games where players have preferences over different types of coalitions (activities) and their sizes instead of other players, which can be expressed as a hedonic game, though. For the case that there is only one type of coalition, Lee and Shoham [29] and Lee and Williams [30] extend this setting such that each player may have a bounded number of friend-enemy relationships to other players.

Goldreich and Ron [22] showed that classic graph problems like connectivity, being Eulerian and cycle freeness are testable with constant query complexity. On the other hand, it is known that testing bipartiteness [24] and expansion [15, 27, 31] have upper and lower bounds of roughly $\Theta(\sqrt{n})$. Turning to more general results, Benjamini et al. [8] proved that every minor-closed property is constant query testable, and Newman and Sohler [32] extended this result to hold for every hyperfinite property. Property testing of annotated (or labelled) graphs has been studied for geometric graphs mainly. For example, testing whether a graph that is embedded into the plane is a Euclidean minimum spanning tree has been studied by Ben-Zwi et al. [7] and Czumaj and Sohler [14]. Ben-Zwi et al. show that any non-adaptive tester has to make $\Omega(\sqrt{n})$ queries, and that any adaptive tester has query complexity $\Omega(n^{1/3})$. Czumaj and Sohler provide a one-sided error tester with query complexity $\tilde{O}(\sqrt{n/\epsilon})$. Hellweg et al. [25] develop a tester for Euclidean $(1+\delta)$-spanners.

Learning hedonic games was studied by Sliwinski and Zick [36]. While property testing focuses on testing whether a game admits a stable outcome or whether an outcome is stable with sublinear complexity, PAC learning constructs a good hypothesis and *PAC stabilization* uses this hypothesis to learn a stable outcome (if possible) using a superlinear number of samples (with possibly linear size). As far as we know, no sublinear algorithms have been developed for hedonic games, yet.

2 PRELIMINARIES

In this paper, we consider undirected graphs with vertex degrees bounded by a constant d. For a graph $G = (V, E)$ at hand, we write $n = |V|$. Without loss of generality, we assume that $V = [n] = \{1, \ldots, n\}$.

2.1 Hedonic Games

A *hedonic game* $(N, \succeq)$ consists of a set of *players* $N = [n]$ and a *preference profile* $\succeq = (\succeq_1, \ldots, \succeq_n)$, where $\succeq_i$ is player i's preference relation over $\mathcal{N}_i = \{C \subseteq N \mid i \in C\}$. A subset $C \subseteq N$ of players is called *coalition*. An output of a hedonic game is a *coalition structure*, i.e., a partition Γ of the player set. Let $\Gamma(i) \in \Gamma$ be the coalition containing $i \in N$. We say that a player i *weakly prefers* a coalition A to a coalition B, if $A \succeq_i B$. Player i *prefers* A to B, denoted by $A \succ_i B$, if $A \succeq_i B$, but $B \not\succeq_i A$; i is *indifferent* between A and B, denoted by $A \sim_i B$, if $A \succeq_i B$ and $B \succeq_i A$.

Since the set $\mathcal{N}_i$ of coalitions a player is contained in, has an exponential size in the number of players, a central question in the study of hedonic games is to define representations that are adequately compact and at the same time as expressive as possible.

One common representation is that of a graph network $G = (N, E)$, where the players are vertices in the graph. In the encoding as defined by Ota et al. [33], for each player $i \in N$, there exists a set $N_i^+ \subseteq N \setminus \{i\}$ of *friends*, set $N_i^- \setminus \{i\}$ of *enemies*, $N_i^+ \cap N_i^- = \emptyset$. The remaining players are considered as *neutral* $N_i^0 = N \setminus (N_i^+ \cup N_i^- \cup \{i\})$. We call this representation FEN-encoding. It can be represented by a labelled graph $G = (N, F \cup E)$ with $F \cap E = \emptyset$, where $j \in N_i^+$ if and only if $(i, j) \in F$, and $j \in N_i^-$ if and only if $(i, j) \in E$. This conforms to the definition of a graphical hedonic game [34] such that a player i's preference of a coalition $C \in \mathcal{N}_i$ over a coalition $D \in \mathcal{N}_i$ only depends on i's neighbourhood $N_i = N_i^+ \cup N_i^-$:

$$C \succeq_i D \iff C \cap N_i \succeq_i D \cap N_i. \tag{1}$$

Here, we extend the players' relations to preferences in the following manner. A value function is specified such that each player $i \in N$ assigns a positive value $f \in \mathbb{R}$ to each $j \in N_i^+$ and a negative value $-e \in \mathbb{R}$ to each $j \in N_i^-$. The corresponding utility function $u_i : \mathcal{N}_i \to \mathbb{R}$, $i \in N$, is defined additively by $u_i(C) = f \cdot |C \cap N_i^+| - e \cdot |C \cap N_i^-|$. For instance, under *friends appreciation* we have $f = d$ and $e = 1$, and under *enemies aversion* this corresponds to $f = 1$ and $e = d$. The preference extension is obtained by $A \succeq_i B \iff u_i(A) \geq u_i(B)$.

Definition 2.1. We call a hedonic game represented by an FEN-encoding with a preference profile extended via a utility function linear in the number of friends and enemies, *FEN-hedonic game*.

Note that *responsiveness* is always satisfied by the considered preference extensions, i.e., $C \cup \{j\} \succ_i C$ and $C \succ_i C \cup \{j'\}$, for each $i \in N$, and each $C \in \mathcal{N}_i$ and $j \in N_i^+$, $j' \in N_i^-$. Since we consider undirected graphs, we obtain symmetric preferences, i.e., $i \in N_j^+$ if and only if $j \in N_i^+$ and $i \in N_j^-$ if and only if $j \in N_i^-$.

Furthermore, we make the following assumptions. We consider graphs of bounded degree $|N_i| \leq d$ represented by an adjacency list; in particular, it can be decided in time independent of the number n of players whether $C \succeq_i D$, and independent of the coalition size $|C|$ and $|D|$. Moreover, it is often useful to restrict the coalition size, e.g., when players are people that have to communicate or when a coalition represents all ads displayed on a single web page. Therefore, we also consider a bounded coalition size of $|C| \leq c$.

By $\mathfrak{G}_n$ we denote the set of graphs with n vertices that represent such a game. The set of coalition structures partitioning n players is denoted by $\mathfrak{C}_n$. For a stability concept, questions of interest are:

- Verification: Given a game and a coalition structure, is it stable?
- Existence: Is a given game stable, i.e., does there exist a stable coalition structure?
- Search: Find a stable coalition structure for a given game.

In the following, let $G = (N, F \cup E)$ be a graph that represents a FEN-hedonic game and let $\Gamma \in \mathfrak{C}_n$ be a coalition structure solving this game. There are several solutions concepts motivated from different perspectives on the game.

On the one hand, Γ is called

perfect if each player $i \in N$ weakly prefers $\Gamma(i)$ to every coalition, i.e., $\Gamma(i) \succeq_i C$ for each $C \in \mathcal{N}_i$, $|C| \leq c$.

This property reflects an ideal situation, but is rather rarely fulfilled. On the other hand, Γ is called

individually rational if for each player $i \in N$, $\Gamma(i)$ is *acceptable*, i.e., $\Gamma(i) \succeq_i \{i\}$.

Individual rationality is guaranteed by $\{\{i\} \mid i \in N\}$.

Other stability notions are based, for example, on the lack of deviations of a single player to another (possibly empty) existing coalition. Let $\mathrm{Fav}(i)$ denote the set of i's favourite coalitions of size at most c, i.e., those coalitions that i weakly prefers over all other coalitions of size at most c. A coalition structure Γ is called

Nash-stable if no player wants to move to another existing or empty coalition, i.e., for each player $i \in N$ and each coalition $C \in \Gamma \cup \{\emptyset\}$ with $|C| < c$, it holds that $\Gamma(i) \succeq_i C \cup \{i\}$;

individually stable if no player can move to another preferred coalition without making a player in the new coalition worse off, i.e., for each player $i \in N$ and for each coalition $C \in \Gamma \cup \{\emptyset\}$ with $|C| < c$, it holds that $\Gamma(i) \succeq_i C \cup \{i\}$ or there exists a player $j \in C$ such that $C \succ_j C \cup \{i\}$;

contractually individually stable if no player can move to another preferred coalition without making a player in the new coalition or in the old coalition worse off, i.e., for each player $i \in N$ and for each coalition $C \in \Gamma \cup \{\emptyset\}$ with $|C| < c$, it holds that $\Gamma(i) \succeq_i C \cup \{i\}$, or there exists a player $j \in C$ such that $C \succ_j C \cup \{i\}$, or there exists a player $j' \in \Gamma(i) \setminus \{i\}$ such that $\Gamma(i) \succ_{j'} \Gamma(i) \setminus \{i\}$.

Note that Nash stability implies individual stability, which, in turn, implies contractual individual stability.

2.2 Graph Property Testing

Let $G = (V, E)$ be a graph with vertex degrees bounded by d and let $\mathcal{P}$ be a graph property, i.e., a set of graphs (e.g., let $\mathcal{P}$ be all graphs that admit a perfect coalition structure). We say that G is *ϵ-far* from a property $\mathcal{P}$ if more than ϵdn edges of G have to be modified in order to convert it into a graph that satisfies the property $\mathcal{P}$, otherwise G is *ϵ-close* to $\mathcal{P}$. A property tester has access to G by querying a function $f_G : V \times [n] \to [n] \cup \{\star\}$, where $f_G(v, i)$ denotes the i^{th} neighbour of v if v has at least i neighbours. Otherwise, $f_G(v, i) = \star$.

Definition 2.2 (One-sided testers). A one-sided error ϵ-tester for a property $\mathcal{P}$ of bounded degree graphs with query complexity q is a randomized algorithm $\mathcal{A}$ that makes q queries to f_G for a graph G. The algorithm $\mathcal{A}$ accepts if G has the property $\mathcal{P}$. If G is ϵ-far from $\mathcal{P}$, then $\mathcal{A}$ rejects with probability at least $2/3$.

A graph property $\mathcal{P}$ is *edge monotone* if for every $G = (V, E) \in \mathcal{P}$, $\{(V, E') \mid E' \subseteq E\} \subseteq \mathcal{P}$. In other words, every subgraph of G is also in $\mathcal{P}$.

3 PROPERTY TESTING OF STABILITY CONCEPTS

To test stability concepts, we generalize the standard edit distance of graph property testing as follows. Since we consider graphs $G = (V, F \cup E)$ that represent FEN-hedonic games, we have to account for the two types of edges: friends and enemies. Therefore, an edge modification is one insertion of an element to or one removal of an element from $F \cup E$, respectively, while maintaining $F \cap E = \emptyset$. In particular, turning a friend edge into an enemy edge is counted as two edge modifications (removing it from F and inserting it into E). The intuition of these semantics is that edge modifications measure the number of compromises that are needed to reach a stable situation. If a partition is too far from being stable, too many compromises are necessary, and the partition should be discarded. Everything in-between is not an ideal situation, but only a few compromises may be affordable.

Now, the existence of a stable outcome in a game is modelled as a graph property as follows.

Definition 3.1 (stability existence property). The set of stable graphs with respect to some stability concept (e.g., Nash stability) is the set of all graphs G that admit a stable coalition structure.

For some stability concepts, the existence of a stable outcome is guaranteed. Nevertheless, the question of whether a given partition Γ satisfies the stability property can still be hard to decide. The worst case time that is needed to verify stability of Γ for all stability concepts mentioned above is at least linear in the number of players. We can, however, tackle the following problem in sublinear time: Given a graph G and a partition of vertices Γ, is Γ a stable outcome for the game represented by G, or is G ϵ-far from being a stable instance for Γ?

Definition 3.2 (Γ-stability verification property). Let $n \in \mathbb{N}$, and let Γ be a partition of $[n]$. The set of Γ-stable graphs with respect to some stability concept (e.g., Nash stability) is the set of n-vertex graphs G such that Γ is a stable coalition structure of G.

Note that, unlike the existence of a stable coalition structure, a stability property is not closed under isomorphism as long as Γ is not permuted additionally. Therefore, extending the basic model of graph property testing to reflect the semantics of hedonic games is the foundation of our main contribution. Access to Γ is provided by a set oracle that supports two queries. A find query returns, given a vertex v, the key of the set that contains v. A member query returns, given a key k and an index i, the i-th element of the set represented by k, or $\star$ if no such element exists.

One beneficial feature of bounded degree graphs is the bounded size of neighbourhoods, i.e., the number of graphs that have constant distance to a given vertex. In hedonic games, this is mirrored by the maximum coalition size, and therefore, we take the coalition size as a parameter into our analysis.

4 PROPERTY TESTING IN THE FEN-MODEL

In this section we study property testers for stability verification problems, resulting in Theorem 1.2, as well as stability existence problems, resulting in Theorem 1.1, for various individual-based stability notions within the previously defined model of FEN-hedonic games.

4.1 Testing Verification Problems

In the following we aim to prove Theorem 1.2, the testability of verification problems with query complexity dependent only on the degree bound, but independent of the graph size or any coalition bound, which is restated in Theorem 4.6 below. In our constructions we relate to the players' favourite coalitions and make use of the following lemma that states that we can easily modify a player's local surroundings to turn the current coalition into a favourite coalition. In other words, only a constant number of compromises suffice to optimise one player's current situation.

Lemma 4.1. *For every graph $G = (N, F \cup E)$ with $|F \cup E|$ bounded by d, each coalition structure Γ of N, it holds that for each $i \in N$, $O(d)$ queries and d edge modifications are sufficient to turn $\Gamma(i)$ into one of i's favourite coalitions in the FEN-hedonic game represented by G.*

Proof. If for player i, it already holds that $\Gamma(i) \in \mathrm{Fav}(i)$, no modification is required. Otherwise we can proceed as follows: Accessing the (at most d) members of N_i requires at most d oracle queries. Moreover, we can ask one oracle query each in order to find out, whether a player $j \in N_i = N_i^+ \cup N_i^-$ is contained in $\Gamma(i)$. For each $j \in \Gamma(i) \cap N_i^-$, delete the edge (i, j) from E; for each $j \in N_i^+ \setminus \Gamma(i)$: remove the edge (i, j) from F. This requires at most $|N_i^+| + |N_i^-| \le d$ edge modifications. Note that this is independent of any bound c of the coalition size. The obtained coalition now only contains friends of i's and i does not have any friends outside of $\Gamma(i)$. Hence, for all considered preference extensions, no coalition is preferred to the current coalition $\Gamma(i)$. □

Many stability concepts are of the form such that stability holds if and only if no player i satisfies a certain condition $\phi(i)$. If there exists a player j that satisfies this condition $\phi(j)$, we call j a *witness* for non-stability.

Next, we observe that due to responsiveness and symmetry in all considered preference extensions, an edge modification that benefits one player, can never be a disadvantage for other players.

Observation 4.2. *Let $i, j \in N$ be two players in the FEN-hedonic game represented by a graph $G = (N, F \cup E)$. Furthermore, let $\succeq_i$ be i's original preference relation, and $\succeq_i'$ the preference relation of i after a modification of edge (i, j), i.e., in the new game represented by $G' = (N, F' \cup E')$ with the same preference extension. The following statements hold:*

(1) If (i, j) is deleted from E, for each $C \in \mathcal{N}_i$, $j \notin C$, it holds that $C \succ_i C \cup \{j\}$ and $C \cup \{j\} \sim_i' C$,
(2) If (i, j) is deleted from F, for each $C \in \mathcal{N}_i$, $j \notin C$, it holds that $C \cup \{j\} \succ_i C$ and $C \cup \{j\} \sim_i' C$,
(3) If (i, j) is deleted from E, and $C \succ_i D$ for two coalitions $C, D \in \mathcal{N}_i$ with $j \in C$, it holds that $C \succ_i' D$.
(4) If (i, j) is deleted from F, and $C \succ_i D$ for two coalitions $C, D \in \mathcal{N}_i$ with $j \notin C$, it holds that $C \succ_i' D$.

The considered stability concepts are defined via properties that need to be avoided for each player. Let ϕ denote such a player property assigning each player i either value 1 (i is a witness against the property) or value 0 (i is not a witness). In the following proofs we require certain conditions to hold for ϕ and show that all considered concepts share these conditions, which enables us to devise a unified testing scheme.

Definition 4.3. Let γ be a stability concept for which there exists a property $\phi : \mathfrak{G}_n \times \mathfrak{C}_n \to \{0, 1\}^n$, $n \in \mathbb{N}$, such that for any game represented by $G = (N, F \cup E)$, $|N| = n$ and a coalition structure $\Gamma \in \mathfrak{C}_n$ it holds that $\phi_i(G, \Gamma) = 0$ for each $i \in N$, if and only if Γ is stable in G with respect to γ. We say that ϕ is *feasible* in our setting if for each n and each $G = (N, F \cup E) \in \mathfrak{G}_n$ and $\Gamma \in \mathfrak{C}_n$ the following conditions are met for each $i \in N$.

(i) $\Gamma(i) \in \mathrm{Fav}(i) \implies \phi_i(G, \Gamma) = 0$.
(ii) The value $\phi_i(G, \Gamma)$ can be determined with a constant number of queries to the oracles for G and Γ (i.e., dependent on ϵ, d, but independent of n).
(iii) If $\phi_i(G, \Gamma) = 0$ and an edge (j, i), $j \notin \Gamma(i)$ is removed from F, resulting in a new game G', it holds that $\phi_i(G', \Gamma) = 0$ remains valid.
(iv) If $\phi_i(G, \Gamma) = 0$ and an edge (j, i), $j \in \Gamma(i)$ is removed from E, resulting in a new game G', it holds that $\phi_i(G', \Gamma) = 0$ remains valid.

We note that if ϕ is feasible, then ϕ is edge monotone.

Lemma 4.4. *Let $G = (N, F \cup E)$ be a graph that represents a FEN-hedonic game and Γ a coalition structure of N. Let, furthermore, γ be a stability concept, for which there exists a feasible player property ϕ. If there are at most k witnesses, $k \cdot d$ edge modifications are sufficient to make the game stable with respect to γ.*

Proof. By Lemma 4.1, for each witness i, d edge modifications are enough to turn $\Gamma(i)$ into a favourite coalition, thus, $\phi(i) = 1$ is no longer satisfied. For each player j that is not a witness, $\phi(j) = 0$ already holds which does not change due to Conditions (iii) and (iv) of Definition 4.3. If there are at most k witnesses, $k \cdot d$ edge modifications are sufficient such that no player satisfies $\phi(i)$, thus, stability with respect to γ holds. □

With the help of this lemma we are now ready to prove that Algorithm 1 provides a property tester for the verification problem of each stability concept with a feasible player property.

Algorithm 1

Require: access to $G = (N, F \cup E)$ and Γ is provided by an oracle, $\phi(G, \Gamma)$ is the corresponding boolean stability function

```
1: function VerificationTester(N, F, E, s)
2:     s ← (1/ε) ln 3
3:     sample s players iid from N
4:     for each sampled player i do
5:         if φ_i(G, Γ) = 1 then
6:             return reject
7:     return accept
```

Theorem 4.5. *Let γ be a stability concept for which there exists a feasible player property ϕ. It holds that Algorithm 1 is a one-sided error property tester for Γ-stability verification with respect to γ.*

Proof. If γ holds, there is no witness for non-stability, i.e., for each sampled vertex $\phi(i) = 0$ holds. Therefore, the tester decides in Line 6 that γ holds with probability 1.

If Γ is ϵ-far from being stable with respect to γ, at least ϵdn edge modifications are required. Thus, by Lemma 4.4, there are at least $\epsilon dn/d = \epsilon n$ witnesses. Hence, the probability that a sampled player is a witness is at least $\epsilon n \cdot 1/n = \epsilon$.

Then, the algorithm correctly rejects if at least one witness is sampled, i.e., the condition in Line 5 is true. The probability of this event is 1 minus the probability that for each sampled player the condition in Line 5 is false, i.e., $\phi_i(G,\Gamma) = 0$. The latter probability is at most

$$(1-\epsilon)^s \leq e^{-(\epsilon \cdot s)} = e^{-\ln 3} = \frac{1}{3}.$$

Thus, the probability that the tester correctly rejects is at least $2/3$.

Since ϕ is feasible, $\phi_i(G,\Gamma)$ can be determined in constant query time. Hence, the tester requires constant query time dependent on the applied function ϕ. □

Now it remains to show that each considered stability property has such a feasible player property ϕ. By Theorem 4.5, we obtain a verification tester by Algorithm 1 with a query complexity depending on ϕ.

Theorem 4.6. *For the FEN-hedonic game model, the* Γ*-stability verification property can be tested with respect to*

(1) perfection and individual rationality with query complexity in $O(d/\epsilon)$*,*
(2) Nash stability, individual and contractual individual stability with query complexity in $O(d/\epsilon)$*.*

Proof. For each considered stability concept we show that there exists a feasible player property such that Theorem 4.5 can be applied. In each case we determine the query complexity of the tester.

perfect: The corresponding player property is

$$\phi_i(G,\Gamma) = 1 \iff \exists C \in \mathcal{N}_i : C \succ_i \Gamma(i).$$

Condition (i) holds by definition of perfection. We have $\phi_i(G,\Gamma) = 1$ if and only if $\Gamma(i)$ does not contain all of i's friends and none of i's enemies, which can be verified in constant query time by asking whether j is in the same coalition as i for each $j \in N_i$. Therefore, Condition (ii) is met with d queries per sampled player. In total, the query complexity is in $O(d/\epsilon)$. Conditions (iii) and (iv) can be implied immediately by Observation 4.2, since the relation $\Gamma(i) \succeq_i C$ remains valid in each relevant case.

individually rational: Here, the player property is

$$\phi_i(G,\Gamma) = 1 \iff \{i\} \succ_i \Gamma(i).$$

Condition (i) holds, since $C \in \mathrm{Fav}(i)$ implies that $C \succeq_i \{i\}$. Since the decision whether $\{i\} \succ_i \Gamma(i)$ only depends on $|\Gamma(i) \cap N_i|$, d queries are sufficient, i.e., a total query complexity in $O(d/\epsilon)$, which satisfies Condition (ii). Again, Conditions (iii) and (iv) can be implied by Observation 4.2.

Nash-stable: A witness i against Nash stability satisfies

$$\phi_i(G,\Gamma) = 1 \iff \exists C \in \Gamma \cup \{\emptyset\} : C \cup \{i\} \succ_i \Gamma(i)$$

Condition (i) holds, since $\Gamma(i) \in \mathrm{Fav}(i)$ implies $\Gamma(i) \succeq_i C$ in particular for C such that $C = C' \cup \{i\}$, $C' \in \Gamma \cup \{\emptyset\}$. We consider the following cases regarding Condition (ii):

(a) If $\Gamma(i) \cap N_i^+ \neq \emptyset$, i wants to deviate to a coalition C with $u_i(C \cup \{i\}) > u_i(\Gamma(i))$. Due to the linearity of the preferences, this can only be $\{i\}$ (with $u_i(\{i\}) = 0$) or a coalition in Γ that contains at least one friend. There are at most $|N_i^+| \leq d$ coalitions in Γ that contain a friend, namely $\Gamma(j)$, $j \in N_i^+$. Hence, at most d comparisons of coalitions are sufficient, which can be done with at most d neighbour and d find queries by Equation (1).
(b) If $\Gamma(i) \cap N_i^+ = \emptyset$, but $N_i^+ \neq \emptyset$, the analysis is analogous to (a).
(c) If $N_i^+ = \emptyset$ and $\Gamma(i) \cap N_i^- = \emptyset$, $\Gamma(i)$ is already one of i's favourite coalitions, hence $\phi_i(G,\Gamma) = 0$.
(d) If $N_i^+ = \emptyset$ and $\Gamma(i) \cap N_i^- \neq \emptyset$, i wants to deviate to the single player coalition $\{i\}$, hence $\phi_i(G,\Gamma) = 1$.

It can be decided with d neighbour queries which of the four cases holds for $\Gamma(i)$. The at most d coalition comparisons require at most d additional find queries. Therefore, $\phi_i(G,\Gamma)$ can be decided with $O(d)$ queries, satisfying Condition (ii). The total query complexity of Algorithm 1 is $O(d/\epsilon)$.
Again, Conditions (iii) and (iv) can be implied by Observation 4.2.

individually stable: A witness i against individual stability satisfies

$$\phi_i(G,\Gamma) = 1 \iff \exists C \in \Gamma \cup \{\emptyset\} : C \cup \{i\} \succ_i \Gamma(i) \text{ and } \forall j \in C : C \cup \{i\} \succeq_j C.$$

Hence, if i is not a witness for a Nash deviation, it cannot be a witness here, either. Therefore, Condition (i) holds. If i wants to deviate, this is due to one of the cases (a), (b), or (d) above. In the latter case, again, $\Gamma(i)$ is not acceptable, and i is always welcome in $\{i\}$. In cases (a) and (b), we have to consider at most $|N_i^+|$ candidate coalitions, i can deviate to. For each neutral player $j \in N_i^0$, it holds that $\Gamma(j) \sim_j \Gamma(j) \cup \{i\}$. Thus, we only have to ask i's neighbours for permission to enter the new coalition, which are in total at most d. In fact, due to the symmetry of preferences, friends always welcome i, and enemies always don't.
We obtain $\phi_i(G,\Gamma) = 0$ if and only if there are no enemies in C, which we can decide with at most d queries. Thus, we can employ the same queries as for Nash stability in order to determine $\phi_i(G,\Gamma)$, which satisfies Condition (ii). The total query complexity of Algorithm 1 is $O(d/\epsilon)$.
If i wants to move to another coalition $C \subseteq \Gamma \cup \{\emptyset\}$ but there exists a player $j \in C$ with $C \succ_j C \cup \{i\}$, then j is i's enemy due to Observation 4.2. Therefore, deleting edges from F cannot make $C \cup \{i\}$ a feasible deviation if it was not feasible before. Thus, Condition (iii) is met. If an edge (i,j) is deleted from E and $j \in \Gamma(i)$, Condition (iv) can only be false if $\Gamma(j) \succ_j \Gamma(j) \cup \{i\}$. However, $i \in \Gamma(j)$ because $\Gamma(j) = \Gamma(i)$, which is a contradiction, and Condition (iv) is met. If i is not a witness because there does not exist any preferred coalition to move to, the arguments for Nash stability can be applied.

contractually individually stable: A witness i against contractual individual stability satisfies $\phi_i(G,\Gamma) = 1$ which holds if and only if there exists some $C \in \Gamma \cup \{\emptyset\}$ such that the two conditions for individual stability ($C \cup \{i\} \succ_i \Gamma(i)$ and $\forall j \in C : C \cup \{i\} \succeq_j C$) and an additional condition ($\forall j' \in \Gamma(i) \setminus \{i\} : \Gamma(i) \setminus \{i\} \succeq_{j'} \Gamma(i)$) hold. Condition (i) holds analogously to individual stability.
Observe that for neutral players $j' \in N_i^0 \cap \Gamma(i)$ it holds that $\Gamma(i) \sim_{j'} \Gamma(i) \setminus \{i\}$ and for enemies $j' \in N_i^-$ it holds that $\Gamma(i) \setminus \{i\} \succ_{j'} \Gamma(i)$. Again, if i wants to deviate to a coalition, cases (a), (b) and (d) remain. In case (a) i has friends in $\Gamma(i)$ that i contractually depends on. Here $\phi_i(G,\Gamma) =$

0. In cases (b) and (d) there are no friends in $\Gamma(i)$, which means there is no contractual dependence. Then, i is a witness against contractual individual stability if and only if it is a witness against individual stability. Thus, in case (d) $\phi_i(G,\Gamma) = 1$ and in case (b) that same queries as above can be applied. Thus, we need at most $O(d)$ queries in order to determine $\phi_i(G,\Gamma)$. The total query complexity of Algorithm 1 is $O(d/\epsilon)$, which satisfies Condition (ii).
Conditions (iii) and (iv) hold with analogous arguments as above. □

4.2 Testing Existence Problems

Now we prove Theorem 1.1. In general, there always exists an individually rational coalition structure. Bogomolnaia and Jackson [9] show that in symmetric additively separable hedonic games there always exists a Nash stable coalition structure. Note that in our model the argument that if a player deviates, the social welfare increases, remains valid, even for bounded coalition size. We provide a sketch of the formal proof for completeness.

Observation 4.7. *Each symmetric FEN-hedonic game (all considered preference extensions) allows a Nash-stable, and consequently individually stable and contractually individually stable coalition structure.*

Proof sketch. Let Γ be a coalition structure containing coalitions $\Gamma(i)$ and $C \subseteq N$ with $|C| \leq c-1$. We assume that $C \cup \{i\} \succ_i \Gamma(i)$. Moreover, let Γ' be the coalition structure obtained if i deviates to C, i.e., Γ' contains $\Gamma(i) \setminus \{i\}$ and $C \cup \{i\}$. The social welfare $\mathrm{SW}(\Gamma)$ of a coalition structure Γ is the sum of all players' utilities of their current coalition. We observe that the difference of the social welfare of Γ' and Γ always increases, which means that there exists a local maximum resulting in a Nash-stable coalition structure. It holds that the difference $\mathrm{SW}(\Gamma') - \mathrm{SW}(\Gamma)$ equals

$$\begin{aligned}
&\sum_{j \in N} \left(u_j(\Gamma'(j)) - u_j(\Gamma(j))\right)\\
&= \underbrace{u_i(C \cup \{i\}) - u_i(\Gamma(i))}_{>0} + \sum_{j \in C \cap N_i} \left(u_j(C \cup \{i\}) - u_j(C)\right)\\
&\quad + \sum_{j \in \Gamma(i) \cap N_i} \left(u_j(\Gamma(i) \setminus \{i\}) - u_j(\Gamma(i))\right)\\
&= \underbrace{u_i(C \cup \{i\}) - u_i(\Gamma(i))}_{>0} + f \cdot |C \cap N_i^+| - e \cdot |C \cap N_i^+|\\
&\quad - f \cdot |\Gamma(i) \cap N_i^+| + e \cdot |\Gamma(i) \cap N_i^+| > 0
\end{aligned}$$

Hence, there always exists a Nash-stable coalition structure, even if the coalition size is bounded. Since Nash stability implies individual and contractual individual stability, they are guaranteed to exist as well. □

There does not necessarily exist a perfect coalition structure. For example, there does not exist any perfect coalition structure for $G = (\{1,2,3\}, F \cup E)$, where $F = ((1,2),(2,3))$, $E = (3,1)$. On the other hand, if $E = \emptyset$, the utility of any coalition structure is at most 0, so singleton coalitions are perfect; if $F = \emptyset$, there exists a perfect coalition structure if and only if no connected component is larger than c. For the general case $|E|, |F| \geq 0$, we show that there exists a tester with one-sided error.

Theorem 4.8. *There is a one-sided error tester with constant query complexity for the existence of a perfect coalition structure in the FEN-hedonic game model with a constant coalition size bound.*

Proof. Let $v \in V$, and observe that C is a favourite coalition of v if and only if $C \cap F(v) = F(v)$ and $C \cap E(v) = \emptyset$. It follows that there exists a perfect coalition structure Γ if and only if there does not exist any edge in E between vertices of the same connected component of $G[V_F]$, where V_F is the set of endpoints in F, i.e., $V_F = \{u \mid (u,v) \in F\}$. This suggests the following algorithm: first, sample a set S of $|S| = 1/\epsilon \ln 3$ vertices at random. For each $v \in S$, we run a BFS that follows only edges in F. If one of these BFS's sees more than c vertices or it discovers two endpoints u, v of the same edge $(u,v) \in E$, the tester rejects. Otherwise it accepts the graph.

By the above observation, every path in G that contains only edges from F must be in the same coalition in a perfect coalition structure. The algorithm rejects only when it finds a path P such that for every coalition structure Γ such that some coalition $C \in \Gamma$ contains P, C also contains $(u,v) \in E$, which is a witness against the existence of a perfect coalition structure.

If G is ϵ-far from having a perfect coalition structure, then at least ϵdn edges in $F \cup E$ have to be removed in order to make G have a perfect coalition structure because having a perfect coalition structure is an edge-monotone property. Let R be a minimal set of edges that have to be removed. Since every vertex is incident to at most d other vertices, at least $2|R|/d > 2\epsilon dn/d > \epsilon n$ vertices must be incident to an edge from $|R|$. Therefore, the probability that none of the vertices in S is incident to an edge in R is at most

$$\left(1 - \frac{\epsilon n}{n}\right)^{\frac{1}{\epsilon}\ln 3} \leq \frac{1}{3},$$

As argued above, if a vertex in S is incident to an edge from R, the tester finds (u,v) and rejects. □

4.3 Extensions to Weighted and Directed Graphs

A natural extension of Definition 2.1 (FEN-hedonic game) is to allow arbitrary weights for edges instead of f for friends and e for enemies. If each edge contributes equally to the edit distance, this does not affect our proofs and Observation 4.7 because they rely on the linearity of the utility function only. If an edge contributes proportional to its weight, we can use the following standard techniques from property testing. Either, we require that the weights are bounded by some value W and we increase the sampling size in our algorithms by a factor of W. To see why this works, imagine an edge with weight w as w parallel edges, which essentially increases the bound on the vertex degrees to $W \cdot d$. On the other hand, unbounded weights cannot be handled in the standard model by constant-query testers because a single edge that has weight $2\epsilon dn$ can make a graph ϵ-far, yet it is almost impossible to find this edge by sampling $O(1)$ vertices uniformly. Therefore, another option is to allow vertex sampling proportional to the weights of incident edges.

Another extension is to consider directed graphs. In property testing, there are two different models of directed graphs. In the bidirectional model, we may see all incoming and outgoing edges. In the unidirectional model, we may see only one type - usually outgoing edges. Although there is a lossy but still sublinear transformation from constant-query testers in the bidirectional model into the unidirectional model [13], the unidirectional model is often much harder to analyse and yields testers with worse query complexity. Nevertheless, assuming bounded out-degree, verification is not affected by the bidirectional model. For the unidirectional model with outgoing edges, the query complexity for individual stability and contractually individual stability depends on (the minimum of) the in-degree or the maximum coalition size: the case analysis changes because we are required to evaluate the preferences of the other members of the (still at most $O(d)$) affected coalitions. Observation 4.7 is not true anymore (see [9] for an example). A perfect coalitions structure exists if the weakly connected components induced by F do not induce any edges from E. Exploring weakly connected components causes no problem in the bidirectional model, but in the unidirectional model it can render exploration almost impossible. For example, consider a cycle of length c with alternating edge direction and all but one edge being friend edges. This graph makes it impossible to form perfect coalitions, and having ϵdn copies of it as subgraphs makes a graph ϵ-far from admitting a perfect coalition structure. The unidirectional model with incoming edges seems to require new techniques even for verification.

5 OPEN QUESTIONS

A natural question that is related to *finding* stable partitions is the following: Given a graph G and a partition Γ, is *the partition* Γ far from being stable in G (instead of the graph being far from Γ-stable)? This can be generalized further: Property testing is a special case of *local computation algorithms* (LCA), where one shall provide oracle access to a solution, given oracle access to the input. In property testing, the solution is a single bit (accept or reject). While it is beyond the scope of sublinear algorithms to actually compute a stable partition, one may seek to develop an LCA that gives oracle access to it.

Generalizing the results we obtained, one may seek to obtain sublinear algorithms for games with unbounded coalition size. Here, the main difficulty is to obtain insights into the local structure of very large, say, linear sized coalitions. Following a slightly different line of thought, one may consider other graphs models like the dense model, where (almost) all players relate to each other and one may ask how two players i, j relate, or the general graph model, where vertices have arbitrary degrees. These models are quite different from the bounded degree model, as well as from each other. For the dense graph model, a characterization using Szemerédi regular partitions is known [1], and it seems possible that coalition formation could be expressed in terms of regular partitions. Much less is known about the general model. So far, most research is still focused on elementary graph problems like counting the number of constant-size cliques [20]. It would be interesting to see whether stability properties are also constant-query testable in these two models. Another direction for further studies is property testers with two-sided error. These testers do not need to provide a witness against the property, but rather *sufficient statistics* that a graph is far from a property with constant probability.

As mentioned in the introduction, there exist also plenty of other stability concepts like (strict) core stability, Pareto-optimality and popularity that can operate on the same preference extension, which may be interesting to analyse in order to obtain a deeper understanding of locality mechanics in FEN-hedonic games. Here, the main difficulty is to circumvent the usually high computational complexity of the exact decision problems.

Finally, one may study other models of hedonic games, in particular with ordinal preferences (e.g., rankings over known edges [28]). This requires further modelling of the oracle access and considered distance measures.

REFERENCES

[1] Noga Alon, Eldar Fischer, Ilan Newman, and Asaf Shapira. 2009. A Combinatorial Characterization of the Testable Graph Properties: It's All About Regularity. *SIAM J. Comput.* 39, 1 (2009), 143–167. https://doi.org/10.1137/060667177

[2] Haris Aziz, Felix Brandt, and Hans-Georg Seedig. 2013. Computing desirable partitions in additively separable hedonic games. *Artificial Intelligence* 195 (2013), 316–334.

[3] Haris Aziz, Paul Harrenstein, Jérôme Lang, and Michael Wooldridge. 2016. Boolean Hedonic Games. In *Proceedings of the Fifteenth International Conference on Principles of Knowledge Representation and Reasoning.* 166–175.

[4] Haris Aziz and Rahul Savani. 2016. Hedonic games. In *Handbook of Computational Social Choice*, F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. Procaccia (Eds.). Cambridge University Press, Chapter 15, 356–376.

[5] Coralio Ballester. 2004. NP-completeness in hedonic games. *Games and Economic Behavior* 49, 1 (2004), 1–30.

[6] Suryapratim Banerjee, Hideo Konishi, and Tayfun Sönmez. 2001. Core in a simple coalition formation game. *Social Choice and Welfare* 18 (2001), 135–153.

[7] Oren Ben-Zwi, Oded Lachish, and Ilan Newman. 2007. Lower Bounds for Testing Euclidean Minimum Spanning Trees. *Inform. Process. Lett.* 102, 6 (2007), 219–225.

[8] Itai Benjamini, Oded Schramm, and Asaf Shapira. 2010. Every Minor-Closed Property of Sparse Graphs Is Testable. *Advances in Mathematics* 223, 6 (2010), 2200–2218. https://doi.org/10.1016/j.aim.2009.10.018

[9] Anna Bogomolnaia and Matthew O. Jackson. 2002. The stability of hedonic coalition structures. *Games and Economic Behavior* 38, 2 (2002), 201–230.

[10] Katarína Cechlárová. 2002. On the complexity of exchange-stable roommates. *Descrete Applied Mathematics* 116 (2002), 279–287.

[11] Katarína Cechlárová and Antonio Romero-Medina. 2001. Stability in coalition formation games. *International Journal of Game Theory* 29 (2001), 487–494.

[12] Georgios Chalkiadakis, Edith Elkind, and Michael J. Wooldridge. 2011. *Computational Aspects of Cooperative Game Theory.* Morgan and Claypool Publishers.

[13] Artur Czumaj, Pan Peng, and Christian Sohler. 2016. Relating Two Property Testing Models for Bounded Degree Directed Graphs. In *Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing (STOC '16).* ACM, New York, NY, USA, 1033–1045. https://doi.org/10.1145/2897518.2897575

[14] Artur Czumaj and Christian Sohler. 2008. Testing Euclidean Minimum Spanning Trees in the Plane. *ACM Transactions on Algorithms (TALG)* 4, 3 (2008), 31.

[15] Artur Czumaj and Christian Sohler. 2010. Testing Expansion in Bounded-Degree Graphs. *Combinatorics, Probability and Computing* 19 (2010), 693–709. Issue Special Issue 5-6. https://doi.org/10.1017/S096354831000012X

[16] Andreas Darmann, Edith Elkind, Sascha Kurz, Jérôme Lang, Joachim Schauer, and Gerhard J. Woeginger. 2018. Group activity selection problem with approval preferences. *Int. J. Game Theory* 47, 3 (2018), 767–796. https://doi.org/10.1007/s00182-017-0596-4

[17] Dinko Dimitrov, Peter Borm, Ruud Hendrickx, and Shao Chin Sung. 2006. Simple priorities and core stability in hedonic games. *Social Choice and Welfare* 26, 2 (2006), 421–433.

[18] J. H. Drèze and J. Greenberg. 1980. Hedonic Coalitions: Optimality and Stability. *Econometrica* 48, 4 (1980), 987–1003.

[19] R.I.M. Dunbar. 1992. Neocortex size as a constraint on group size in primates. *Journal of Human Evolution* 22, 6 (1992), 469 – 493. https://doi.org/10.1016/0047-2484(92)90081-J

[20] Talya Eden, Dana Ron, and C. Seshadhri. 2018. On Approximating the Number of K-Cliques in Sublinear Time. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (STOC 2018).* ACM, New York, NY, USA, 722–734. https://doi.org/10.1145/3188745.3188810

[21] Edith Elkind and Michael Wooldridge. 2009. Hedonic coalition nets. In *Proceedings of the Eighth International Conference on Autonomous Agents and Multiagent Systems.* 417–424.

[22] Goldreich and Ron. 2002. Property Testing in Bounded Degree Graphs. *Algorithmica* 32, 2 (2002), 302–343. https://doi.org/10.1007/s00453-001-0078-7

[23] Oded Goldreich. 2017. *Introduction to Property Testing*. Cambridge University Press.

[24] Oded Goldreich and Dana Ron. 1999. A Sublinear Bipartiteness Tester for Bounded Degree Graphs. *Combinatorica* 19, 3 (1999), 335–373. https://doi.org/10.1007/s004930050060

[25] Frank Hellweg, Melanie Schmidt, and Christian Sohler. 2010. Testing Euclidean Spanners. *Property Testing* 6390 (2010), 306–311.

[26] Ayumi Igarashi and Edith Elkind. 2016. Hedonic games with graph-restricted communication. In *Proceedings of the Fifteenth International Conference on Autonomous Agents and Multiagent Systems*. 242–250.

[27] Satyen Kale and C. Seshadhri. 2011. An Expansion Tester for Bounded Degree Graphs. *SIAM J. Comput.* 40, 3 (2011), 709–720. https://doi.org/10.1137/100802980

[28] Jérôme Lang, Anja Rey, Jörg Rothe, Hilmar Schadrack, and Lena Schend. 2015. Representing and solving hedonic games with ordinal preferences and thresholds. In *Proceedings of the Fourteenth International Conference on Autonomous Agents and Multiagent Systems*. 1229–1237.

[29] Hooyeon Lee and Yoav Shoham. 2015. Stable Invitations. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, January 25-30, 2015, Austin, Texas, USA*. 965–971. http://www.aaai.org/ocs/index.php/AAAI/AAAI15/paper/view/9368

[30] Hooyeon Lee and Vassilevska Williams. 2017. Complexity of the Stable Invitations Problem. In *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence, February 4-9, 2017, San Francisco, California, USA*. 579–585. http://aaai.org/ocs/index.php/AAAI/AAAI17/paper/view/14278

[31] Asaf Nachmias and Asaf Shapira. 2010. Testing the Expansion of a Graph. *Information and Computation* 208, 4 (2010), 309–314. https://doi.org/10.1016/j.ic.2009.09.002

[32] Ilan Newman and Christian Sohler. 2013. Every Property of Hyperfinite Graphs Is Testable. *SIAM J. Comput.* 42, 3 (2013), 1095–1112. https://doi.org/10.1137/120890946

[33] Kazunori Ota, Nathanaël Barrot, Anisse Ismaili, Yuko Sakurai, and Makoto Yokoo. 2017. Core Stability in Hedonic Games among Friends and Enemies: Impact of Neutrals. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence*. 359–365.

[34] Dominik Peters. 2016. Grahical Hedonic Games of Bounded Treewidth. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence*. 586–593.

[35] Dominik Peters and Edith Elkind. 2015. Simple causes of complexity in hedonic games. In *Proceedings of the Twentyfourth International Joint Conference on Artificial Intelligence*. 617–623.

[36] Jakub Sliwinski and Yair Zick. 2017. Learning Hedonic Games. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence* (2017) *(IJCAI'17)*. AAAI Press, 2730–2736. http://dl.acm.org/citation.cfm?id=3172077.3172269

[37] Shao Chin Sung and Dinko Dimitrov. 2010. Computational complexity in additive hedonic games. *European Journal of Operational Research* 203, 3 (2010), 635–639. https://doi.org/10.1016/j.ejor.2009.09.004

[38] Gerhard J. Woeginger. 2013. A Hardness Result for Core Stability in Additive Hedonic Games. *Mathematical Social Sciences* 65, 2 (2013), 101–104.