

k -Prize Weighted Voting Games

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ABSTRACT

We introduce a natural variant of weighted voting games, which we refer to as k -Prize Weighted Voting Games. Such games consist of n players with weights, and k prizes, of possibly differing values. The players form coalitions, and the i -th largest coalition (by the sum of weights of its members) wins the i -th largest prize, which is then shared among its members. We present four solution concepts to analyse the games in this class, and characterise the existence of stable outcomes in games with three players and two prizes, and in games with uniform prizes. We then explore the efficiency of stable outcomes in terms of Pareto optimality and utilitarian social welfare. Finally, we study the computational complexity of finding stable outcomes.

KEYWORDS

Coalition formation; Cooperative game theory; Cooperative games with externalities; Core; Efficiency; Partition function form; Stability; Weighted Voting Game.

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1 INTRODUCTION

Weighted voting games (WVGs) are an elegant and practically important class of simple cooperative games, which have been extensively studied in the multi-agent systems community [6, 10]. In a WVG, each player has a numeric weight, and the game is played by the players forming coalitions. The weight of a coalition is defined as the sum of the weights of its members, and a coalition's value is 1, if the coalition's weight exceeds a predefined threshold

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$q > 0$, and is 0 otherwise. In this way, a coalition is said to be *winning* if the total weight of all its members exceeds the static threshold q . WVGs are usually designed so that there can be just one winning coalition: simple majority games, where each player has a weight of 1, and a coalition is winning if it contains a majority of the players, are an obvious example.

However, there are settings that can be modelled by WVGs with more than one winning coalition. For example, political parties may form coalitions after an election, which then compete to form the Government (1st prize) or the Opposition (2nd prize). This observation motivates us to consider a variation of WVGs in which there is more than one prize, and the criterion for winning a prize is determined by the weight of a coalition *relative* to others. We refer to these new games as *k -Prize Weighted Voting Games*. The idea of such games is very intuitive: players form coalitions, and the coalition with the largest cumulative weight takes first prize, the coalition with the second largest weight takes second prize, and so on, until all k prizes have been allocated or all coalitions have been considered. Note that all coalitions that do not rank among the k heaviest ones, if any, receive a zero payoff.

It is important to note that our model permits *externalities*: unlike in conventional WVGs, the performance of a coalition in a k -prize game depends not just on its own makeup, but also on the other coalitions that form. Formally, k -Prize Weighted Voting Games belong to a class of games known as *partition function games* (PFGs), where the value of a coalition depends both on the coalition's members as well as the coalitions that other players form [23–25].

The remainder of this paper is structured as follows: In Section 2, we formally define the game and an outcome of the game. We then define four deviation concepts along with their corresponding stability concepts in Section 3. In Section 4 we provide necessary and sufficient conditions for the existence of stable outcomes in two sub-classes of games: games with three players and two prizes, and games with uniform prizes. In Section 5, we explore the efficiency of stable outcomes in terms of Pareto optimality and utilitarian social welfare. Section 6 focuses on the computational complexity of stability-related problems, such as finding a beneficial deviation, deciding stability of a given outcome in a game, and determining

the existence of stable outcomes in a given game. Finally, we survey related work in Section 7 and conclude in Section 8.

2 PRELIMINARIES

A *k-Prize Weighted Voting Game* consists of n players with weights, and k prizes, where $k \leq n$, of possibly differing positive rational values. Players form coalitions, and the i -th largest coalition (by the sum of weights of its members) wins the i -th largest prize, which is shared among its members. If there are more than k coalitions, then coalitions that are not among the k largest coalitions do not receive a prize. A tie-break relation $>$ determines which coalition wins a given prize in the case where multiple coalitions have the same weight. Formally, the game is defined as follows:

Definition 2.1 (k-WVG). A *k-Prize Weighted Voting Game (k-WVG)* or also simply *game* is a tuple $G = (N, \vec{w}, \vec{p}, >)$, where

- $N = \{1, \dots, n\}$ is the set of n players;
- $\vec{w} = (w_1, \dots, w_n) \in \mathbb{Q}_+^n$ is a vector of *weights*, where w_i is the weight of player i . We use $w(S) := \sum_{i \in S} w_i$ to denote the sum of weights of a set of players $S \subseteq N$;
- $\vec{p} = (p_1, \dots, p_k) \in \mathbb{Q}_+^k$ is a vector of positive *prize values* in descending order, where $k \leq n$.
- $>$ is a strict total order on the powerset of all players 2^N , where given two sets $A, B \subseteq N$, $A > B$ means that $w(A) > w(B)$, or $w(A) = w(B)$ and A wins the tie-break against B .

Subsuming N and $>$, we also denote $G = (N, \vec{w}, \vec{p}, >)$ by $(\vec{w}; \vec{p})$.

Players form a partition $\pi \in \Pi(N)$ of N , where $\Pi(X)$ represents the set of all partitions of a finite set X into pairwise disjoint subsets. The *value* $v(C, \pi)$ of a coalition C within a partition π is the value of the prize that is won by C . Formally, we have

$$v(C, \pi) = \begin{cases} p_i & \text{if } C \text{ is the } i^{\text{th}} \text{ largest coalition in } \pi \text{ w.r.t. } > \\ 0 & \text{otherwise.} \end{cases}$$

When we refer to the largest or the smallest player/coalition, we implicitly use $>$ to break ties. In all computational complexity issues that we consider, we assume access to a polynomial time oracle that can decide, given two coalitions C_1 and C_2 , whether $C_1 > C_2$ or $C_2 > C_1$. We say that a coalition C in a coalition structure π is *winning* if it wins one of the k prizes, i.e., $v(C, \pi) > 0$.

In the remainder of the paper, we refer to *k-WVGs* as simply *games*, and we use the notation $(\vec{w}; \vec{p})$ to represent a game, where it is assumed that $N = \{1, \dots, |\vec{w}|\}$, and that some $>$ is given.

Definition 2.2 (Outcome). An *outcome* of a *k-WVG* is a tuple $\omega = (\pi, \vec{x})$, where:

- $\pi = \{C_1, \dots, C_m\}$ is a partition of N into coalitions.
- $\vec{x} = (x_1, \dots, x_n)$ is the vector of payoffs, where x_i is the payoff to player i , such that $x_i \geq 0$ for all $i \in N$ and $\sum_{i \in C_j} x_i = v(C_j, \pi)$ for all $j \in \{1, \dots, m\}$. Given a subset of players $S \subseteq N$, we write $x(S)$ to denote the sum of the payoffs to these players, i.e., $x(S) := \sum_{i \in S} x_i$.

We write Ω for the set of all possible outcomes of a game. Where there is no ambiguity, we may use the shorthand ' $i \dots k$ ' to present the coalition $\{i, \dots, k\}$ in a partition, and ' $|$ ' to separate coalitions within the partition, enclosed by square brackets. For example, the partition of three players $\{\{1, 2\}, \{3\}\}$ can be written as $[1 \ 2 \ | \ 3]$.

Conventionally, we order the coalitions within a partition according to $>$, so we can easily identify the coalition winning a certain prize.

Example 2.3 (A k-WVG and an outcome). Consider a game of 4 players and 2 prizes: $G = (\vec{w}; \vec{p})$ where $\vec{w} = (0.45, 0.25, 0.24, 0.06)$ and $\vec{p} = (2, 1)$, and an outcome $\omega = (\pi, \vec{x})$ where $\pi = [2 \ 3 \ | \ 1 \ | \ 4]$ and $\vec{x} = (1, 0.8, 1.2, 0)$. In this outcome, players 2 and 3 form a coalition to take the first prize: $v(\{2, 3\}, [2 \ 3 \ | \ 1 \ | \ 4]) = p_1 = 2$. They split the prize as $(x_2, x_3) = (0.8, 1.2)$. Player 1 forms a singleton coalition, i.e., a coalition consisting of only one player, and claims the second prize of value $v(\{1\}, [2 \ 3 \ | \ 1 \ | \ 4]) = p_2 = 1$ by itself. Player 4 is left without a prize and receives a payoff of 0.

Note that the outcome in Example 2.3 is intuitively 'unstable' for a number of reasons: First, player 1 can form a coalition with player 2 or player 4 (or both), so as to be guaranteed to win first prize ($p_1 = 2$), which is greater than the sum of their current payoffs ($x_1 + x_2 = x_1 + x_2 + x_4 = 1.8$ and $x_1 + x_4 = 1$). Similarly, players 2 and 4 can together secure second prize ($p_2 = 1$), which is greater than the sum of their current payoffs ($x_2 + x_4 = 0.8$). Or perhaps player 2 could 'go at it alone' and try to secure second prize ($p_2 = 1 > x_2 = 0.8$), depending on whether players 3 and 4 would form a coalition and also compete for the second prize? To reason about stability, we must first formalise the concept of a deviation.

3 DEVIATION AND STABILITY CONCEPTS

We introduce four deviation concepts and their corresponding notions of stability. These concepts differ by what the deviating players are permitted to do: whether they form a single coalition, and, in the case that they split up into multiple deviating coalitions, the incentive requirements that must be satisfied. In all our deviation concepts, the deviating players are *maximally pessimistic* regarding the actions of the *residual* (non-deviating) players, i.e., they assume that the residual players would partition themselves in order to prevent the deviation from being successful if possible.

Definition 3.1 (SCD). A *single-coalition deviation (SCD)* from an outcome (π, \vec{x}) is a set $C \subseteq N$ such that for all $\pi' \in \Pi(N)$ with $C \in \pi'$, we have

$$\sum_{i \in C} x_i < v(C, \pi').$$

Definition 3.2 (MCD). A *multi-coalition deviation (MCD)* from an outcome (π, \vec{x}) is a partition $\pi_D \in \Pi(D)$ of a set $D \subseteq N$ such that for all $\pi' \in \Pi(N)$ with $\pi_D \subseteq \pi'$ and all $C \in \pi_D$, we have

$$\sum_{i \in C} x_i < v(C, \pi').$$

Definition 3.3 (wMCD). A *weak multi-coalition deviation (wMCD)* from an outcome (π, \vec{x}) is a partition $\pi_D \in \Pi(D)$ of a set $D \subseteq N$ such that for all $\pi' \in \Pi(N)$ with $\pi_D \subseteq \pi'$, it holds that:

- (1) for all $C \in \pi_D$ we have

$$\sum_{i \in C} x_i \leq v(C, \pi');$$

- (2) for some $C \in \pi_D$, the inequality in (1) is strict.

Definition 3.4 (MCDT). A *multi-coalition deviation with inter-coalition transfer (MCDT)* from an outcome (π, \vec{x}) is a partition

$\pi_D \in \Pi(D)$ of a set $D \subseteq N$ such that for all $\pi' \in \Pi(N)$ with $\pi_D \subseteq \pi'$, we have

$$\sum_{i \in D} x_i < \sum_{C \in \pi_D} v(C, \pi').$$

For a deviation concept $\theta \in \{SCD, MCD, wMCD, MCDT\}$, we say that an outcome (π, \vec{x}) is *susceptible to a θ* if there exists a θ from (π, \vec{x}) . With these concepts defined, we can introduce the notion of stability with respect to each deviation concept.

Definition 3.5 (Stability and Core). Given any of the above deviation concepts $\theta \in \{SCD, MCD, wMCD, MCDT\}$, let $\theta(G) \subseteq \Omega$ denote the set of outcomes that are susceptible to a θ in the game G . We say an outcome is *θ -stable* if it is not susceptible to a θ . The set of outcomes that are θ -stable is the *θ -core* of the game, i.e., $\theta\text{-core}(G) = \Omega \setminus \theta(G)$.

It is worth noting that the concept of the SCD-core coincides with that of α -core [2] in the more general partition form games.

REMARK 1 (IRRELEVANCE OF COALITION STRUCTURE). For a given outcome (π, \vec{x}) , observe that no consideration is given to the coalition structure π when assessing whether it is susceptible to some deviation concept; Only the payoff vector \vec{x} is relevant. It follows that two outcomes with identical payoff vectors must be susceptible to the same deviation concepts, and have the same stability properties. Formally, given any two outcomes $\omega = (\pi, \vec{x})$, $\omega' = (\pi', \vec{x}')$ where $\vec{x} = \vec{x}'$, ω is stable to some deviation concept if and only if ω' is stable to the same deviation concept. We can therefore also say whether a payoff vector \vec{x} is susceptible (or stable) to some deviation concept.

Now, we analyse the relationship between the different deviation concepts. The following relations are immediate:

REMARK 2 (HIERARCHY OF SOLUTION CONCEPTS). For every game k -WVG G it holds that

$$SCD(G) \subseteq MCD(G) \subseteq wMCD(G) \subseteq MCDT(G).$$

Consequently, we have

$$MCDT\text{-core}(G) \subseteq wMCD\text{-core}(G) \subseteq MCD\text{-core}(G) \subseteq SCD\text{-core}(G).$$

In summary, in an SCD, all deviating players form a single deviating coalition; an MCD allows the deviating players to form multiple deviating coalitions (i.e., a deviating partition) so that each deviating coalition must be strictly better off; a wMCD is similar to an MCD, except only one deviating coalition needs to be strictly better off, while others are weakly better off; an MCDT requires only that the total value of all deviating coalitions exceeds the current payoff of all deviating players, with the implicit assumption that coalitions can make transfers to one another to ensure that all deviating players are strictly better off.

Intuitively, we can think of an MCD as multiple deviating coalitions ‘colluding’ with each other, so that each deviating coalition is better off and does not try to prevent the other deviators from succeeding (as the residual players are assumed to do). A wMCD is, in a sense, a ‘hopeful’ MCD, where the deviating coalitions that are strictly better off (the ‘MCD’ component) rely on the other deviating coalitions to be content with their payoff (i.e., $v(C, \pi') = x(C)$) and not act against them. In some cases, there is another interpretation that does not rely on the good will of some deviating coalitions. If,

depending on the residual players’ response, each of the deviating coalitions has some chance of being strictly better off, then each of the deviating coalitions has the incentive to form a wMCD.

We now show that SCD, MCD, wMCD and MCDT are, in fact, distinct concepts.¹

PROPOSITION 3.6. For some k -WVG G , $SCD(G) \subsetneq MCD(G)$.

PROOF. By definition, $SCD(G) \subseteq MCD(G)$ for each G . We show that there exists a game G and an outcome ω such that $\omega \in MCD(G)$ and $\omega \notin SCD(G)$, through the use of tie-breaking.

Consider a game with six players $N = \{1, \dots, 6\}$ all with weight 1 and two prizes $\vec{p} = (3, 2)$. Assume a tie-breaking order such that, for any two distinct coalitions A and B both containing exactly two members, we have $A > B > \{3, 4\}$ whenever $6 \in B \setminus A$ and $A \neq \{3, 4\}$, i.e., A beats B if B contains 6 and A does not, unless $A = \{3, 4\}$.

Consider the outcome $\omega = (\pi, \vec{x})$ with $\pi = [1\ 2\ 3 \mid 4\ 5 \mid 6]$ and $\vec{x} = (1, 1, 1, 1, 1, 0)$. Let $X = \{1, 2\}$ and $Y = \{5, 6\}$. We find that $[X|Y]$ constitutes an MCD. To see this, note that $x(X) = 2 < p_1$ and $x(Y) = 1 < p_2$, that is, X would prefer p_1 and Y would prefer p_2 compared to their respective payoffs in ω . Observe that $N \setminus (X \cup Y) = \{3, 4\}$. Accordingly, $X > Y > \{3, 4\}$, so players 3 and 4 cannot prevent this deviation by either staying separate or forming a coalition. Thus, ω is susceptible to an MCD by $[X|Y]$.

However, ω is not susceptible to an SCD. To see this, first observe that every coalition A that would prefer p_1 to its current payoff has either (1) $|A| = 3$ and $6 \in A$, or (2) $|A| \leq 2$ (as otherwise $x(A) \geq 3$). Let $B = N \setminus A$. In case (1) we have $|B| = 3$ and $6 \notin B$. Hence, $B > A$. In case (2) we have $w(B) \geq 4 > 2 = w(A)$. In either case, A cannot be guaranteed to win first prize. On the other hand, we note that for every coalition A that would prefer p_2 to its current payoff, i.e., $x(A) < p_2$, either (1) $|A| = 1$ or (2) $|A| = 2$, $6 \in A$. In case (1), observe that $|N \setminus A| = 5$ and we can partition $N \setminus A$ into sets B and B' such that $|B| = 3$ and $|B'| = 2$. Then, $B > B' > A$. In case (2), we can partition $N \setminus A$ into sets B and B' , where $|B| = |B'| = 2$ and $B, B' \neq \{3, 4\}$. Thus, we have $B > B' > A$. In either case, we find that A cannot be guaranteed to win second prize. The outcome ω is therefore not susceptible to an SCD. \square

PROPOSITION 3.7. For some k -WVG G , $MCD(G) \subsetneq wMCD(G)$.

PROOF. Consider the game $G = ((0.4, 0.35, 0.25), (2, 1))$ and the outcome $\omega = ([2\ 3 \mid 1], (1, 1, 1))$. It can be verified that ω is MCD-stable but not wMCD-stable (see Proposition 4.1 case 2). \square

PROPOSITION 3.8. For some k -WVG G , $wMCD(G) \subsetneq MCDT(G)$.

PROOF. Since $wMCD(G) \subseteq MCDT(G)$ by definition, we show that there exists a game G and an outcome ω such that $\omega \in MCDT(G)$ and $\omega \notin wMCD(G)$.

Consider the 4-player, 3-prize game $G = ((7, 4, 4, 2); (2, 1.1, 1.1))$ and the outcome $\omega = ([1 \mid 2 \mid 3 \mid 4], (2, 1.1, 1.1, 0))$. We show that ω is susceptible to an MCDT. Consider the deviating partition $[2\ 3 \mid 4]$: the only coalition structure that can be induced by the deviating partition is $([2\ 3 \mid 1 \mid 4])$, where the deviating coalition would earn the 1st and 3rd prize ($2 + 1.1 = 3.1$), exceeding their current payoff ($x(\{2, 3, 4\}) = 2.2$). Note that this particular deviation is not

¹Note that Proposition 3.6 depends on a specifically chosen tie-breaking order.

a wMCD, since players 2 and 3 would have no incentive to deviate ($x(\{2, 3\}) = 2.2 > 2 = p_1$) without some transfer from player 4.

Suppose for contradiction that ω is also susceptible to a wMCD, for some deviating coalition $D \subseteq N$. It cannot be the case that $|D| = 4$ (i.e., everyone jointly deviates), because all prizes are claimed, and hence no deviation can lead to a strict improvement for some deviating coalition and no loss to others. Also, it cannot be the case that $|D| = 1$, since each player already forms a singleton coalition.

Consider the case $|D| = 2$: D cannot form two singleton deviating coalitions, since they could do no better than the existing outcome; For D to form a single deviating coalition of two members, the coalition must be able to achieve a prize that is strictly better than its current payoff. Clearly, any pair involving player 1 can do no better (player 1 already wins the largest prize), nor does any pair involving player 4, since the other player can do no better. Finally, the pair $(2, 3)$, who can form a coalition larger than $\{1\}$, would not benefit by doing so, as the first prize is smaller than their current joint payoffs under ω .

Finally, consider the case $|D| = 3$: The deviating players cannot form a partition of the form $[i \ j \ k]$ because there is no single prize that exceeds any three players' combined payoff, nor can they form $[i \ j \ | \ k]$ because such a deviation would result in the same outcome. We are left with the case of $[i \ j \ | \ k]$. From the case $|D| = 2$, we saw that no pair $[i \ j]$ could benefit from a deviation, and yet no singleton $[k]$ could strictly improve their payoff. Therefore, there are no feasible deviations with $|D| = 3$. Hence, ω is not susceptible to a wMCD and is susceptible to an MCDT. \square

4 CHARACTERISING STABILITY

Before we explore the computational complexity of finding stable outcomes in general k -WVGs, we show that for two classes of k -WVGs this is relatively straightforward.

3-player 2-prize games: We first consider the special class of games with 3-players and 2-prizes and show that we can easily decide whether such games have a stable outcome, and identify one such outcome. Then, without loss of generality, we normalise the prize values in such games to $(R, 1)$, where $R > 1$.

PROPOSITION 4.1. *A k -WVG with 3 players and 2 prizes, where players are labelled in descending order according to $>$, has an SCD/MCD-stable outcome if and only if $\{1\} > \{2, 3\}$, or $\{2, 3\} > \{1\}$ and $R \leq 2$. In the first case, an SCD/MCD-stable outcome must have payoff $\vec{x} = (R, 1, 0)$; in the second case an SCD/MCD-stable outcome must have payoff $\vec{x} = (1, 1, R - 1)$.*

A game has a wMCD/MCDT-stable outcome if and only if $\{1\} > \{2, 3\}$, and a wMCD/MCDT-stable outcome must have payoff $\vec{x} = (R, 1, 0)$.

SKETCH OF PROOF. We provide an outline of the proof, consisting of all the cases that need to be considered.²

Firstly, observe that there are only two tie-breaking orders consistent with $\{1\} > \{2\} > \{3\}$, namely, one with $\{1\} > \{2, 3\}$ and one with $\{2, 3\} > \{1\}$.

The first case we consider is when $\{1\} > \{2, 3\}$. For this, it suffices to show that: (i) $\vec{x} = (R, 1, 0)$ is MCDT-stable and (ii) $\vec{x} = (R, 1, 0)$

Table 1: Payoff vector in the core of 3-player 2-prize games for each case. \emptyset signifies an empty core.

Case	SCD/ MCD-core	wMCD/ MCDT-core
$\{1\} > \{2, 3\}$	$(R, 1, 0)$	$(R, 1, 0)$
$\{2, 3\} > \{1\}, R \leq 2$	$(1, 1, R - 1)$	\emptyset
$\{2, 3\} > \{1\}, R > 2$	\emptyset	\emptyset

is the only SCD-stable payoff vector. The next case to consider is when $\{2, 3\} > \{1\}$ and $1 < R \leq 2$. In this case, we can establish that: (i) $\vec{x} = (1, 1, R - 1)$ is MCD-stable; (ii) $\vec{x} = (1, 1, R - 1)$ is not wMCD-stable; and (iii) $\vec{x} = (1, 1, R - 1)$ is the only SCD-stable payoff. Finally, the last case is when $\{2, 3\} > \{1\}$ and $R > 2$. In this scenario, a simple contradiction argument suffices to show that no payoff vector \vec{x} is SCD-stable.

Exploiting Remark 2, these findings give us our result, which is also summarised in Table 1. \square

REMARK 3 (EMPTINESS OF THE CORE). *The results above (summarised in Table 1) illustrate that the core of a game, in relation to any deviation concept, can be empty.*

Uniform-prize games: We now consider a special class of k -WVG where all prizes are of equal value (normalised to 1), which we refer to as k -uniform-prize WVGs. We show that for this special class of games, deciding whether a game has an SCD-stable outcome can be easily done via a recursive argument. The following definitions for uniform-prize games establish the necessary concepts for our main result.

Definition 4.2 (Singleton winner). A player i is a *singleton winner* of a k -uniform-prize WVG if it can win a prize as a singleton coalition, regardless of the coalition structure of the remaining players, i.e., for all $\pi \in \{\pi' \in \Pi(N) \mid \{i\} \in \pi'\}$, we have $v(\{i\}, \pi) = 1$.

LEMMA 4.3. *For any k -uniform-prize WVG, if $n = k$, then all players are singleton winners; if $n > k$, a player $i \in N$ is a singleton winner if and only if $w_i > w^*$ where $w^* = \frac{1}{k+1}w(N)$, or $w_i = w^*$ and for every partition π of the players $N \setminus \{i\}$ into k coalitions, there is a coalition $C_j \in \pi$ such that $\{i\} > C_j$.*

PROOF. If $n = k$, then at most k coalitions can be formed by all the players, and therefore each player is guaranteed a prize. If $n > k$, then a player i can secure one of the k prizes as a singleton if $w_i > w^*$, because the collective weight of the remaining players is $w(N \setminus \{i\}) = w(N) - w_i < w(N) \frac{k}{k+1} = kw^*$ and cannot form k coalitions each of size at least w^* . If instead $w_i = w^*$, then $w(N \setminus \{i\}) = kw^*$ and it may be possible for the remaining players to form exactly k coalitions each of size w^* . In this case, the singleton player i can win one of the k prizes if and only if for all partitions of $N \setminus \{i\}$ into k coalitions of weight w^* , the singleton beats (on tie-break) at least one of the k coalitions in the partition. \square

Definition 4.4 (Reduction & Irreducible game). Let s be the set of singleton winners of a k -uniform-prize WVG G . A *reduction* of G is the removal of all singleton winners, and the removal of the same number of prizes as there are singleton winners, from the game. If

²The full proof is provided in Appendix A of the full paper, which can be found at <https://arxiv.org/abs/2302.13888>

$k > |s|$, the reduction of a game G results in a uniform-prize game G' with $n - |s|$ players and $k - |s|$ prizes, with the remaining players having the same weight as in G ; if $k = |s|$, then no prizes are left after the reduction and there is no reduced game. A game with no singleton winner is *irreducible*.

COROLLARY 4.5. *Let $w^* = \frac{1}{k+1}w(N)$. In an irreducible game, either i each player has weight $< w^*$, or ii one player i has weight $w_i = w^*$, all remaining players each have weight $< w^*$, and the remaining players can form k coalitions each of weight w^* and beat $\{i\}$ on tie-break.*

Definition 4.6 (ISW). An *iterative singleton winner (ISW)* of a game G is one of the singleton winners removed in the iterative reduction of G until the game cannot be further reduced.

Inspired by the concept of maximin share as it occurs in the literature on fair allocations (see, e.g., [5]), we now define the notion of an MMS partition and an MMS set.

Definition 4.7 (MMS partition & MMS set). A partition $\pi \in \Pi(N)$ is a *maximin share (MMS) partition* of players N into l sets if it maximises the weight of the smallest set with respect to $>$. Because $>$ is a strict total order, an *MMS partition* always exists, and all MMS partitions have the same smallest set, which we refer to as the *MMS set* or $MMS(N, l)$.

Observe that for any set of players S within an MMS partition that is not the smallest (i.e., $S \in \pi \setminus MMS(N, l)$), we have $MMS(N, l) = MMS(N \setminus S, l - 1)$.

LEMMA 4.8. *The SCD-core of an irreducible game is empty.*

PROOF. Let G be an irreducible game with n players and k prizes, and let the ‘winning threshold’ $w^* = \frac{1}{k+1}w(N)$. We show that we can find an SCD in any outcome $\omega = (\pi, \vec{x})$. By Remark 1, we need only to consider the payoff vector \vec{x} of an outcome and not its coalition structure π .

Consider an MMS partition ϕ of N into $k + 1$ sets. Label the $k + 1$ sets of players as S_1, \dots, S_{k+1} in descending order, i.e., $S_a > S_b$ for all $a < b$, and $S_{k+1} = MMS(N, k + 1)$. Note that S_{k+1} cannot be empty since $n \geq k + 1$ in an irreducible game. We now construct an SCD by considering each S_d where $d \in \{1, \dots, k\}$ (i.e. excluding the smallest set S_{k+1}).

Suppose there exists d such that $x(S_d) < 1$, then S_d can form an SCD. To see this, note that $S_{k+1} = MMS(N, k+1) = MMS(N \setminus S_d, k)$, and that $S_d > S_{k+1}$. This means that under any partition of $N \setminus S_d$ into k sets, S_d will always beat the smallest set and win one of the k prizes. S_d can therefore guarantee a prize value of $1 > x(S_d)$ by deviating.

If there does not exist d such that $x(S_d) < 1$, then it must be the case that $x(S_d) = 1$ for all $d \in \{1, \dots, k\}$ and $x(S_{k+1}) = 0$, because the total prize value to be shared amongst all players is exactly k . We consider two cases of the distribution of \vec{x} . First is the case where $x_i \in \{0, 1\}$ for all $i \in N$, i.e., the payoff to each player is either 1 or 0. This can occur either because all winning coalitions are singletons, or because some winning coalitions contain all but one zero-payoff players. Since there are k prizes, there are exactly k players with payoff 1, and by Corollary 4.5, either a) each winning player has weight less than w^* , or b) at most one has weight w^* but loses on

all tie-breaks. If $k > 1$, the total weight of all zero-payoff players must exceed $w(N) - kw^* = w^*$, and these players can guarantee to win a prize by forming an SCD. If $k = 1$ and the winning player has weight w^* , then by Corollary 4.5 the zero-payoff players can also form an SCD of size w^* and take the prize from the winning player.

Now consider the case where $x_i \in (0, 1)$ for some $i \in N$, i.e., there is some set S_d in which at least 2 players have a positive payoff. Let the smallest of these players be player i with weight w_i . We argue that $S_d \cup S_{k+1} \setminus \{i\}$ can form an SCD. First note that because $x_i > 0$, $S_d \setminus \{i\}$ can incentivise S_{k+1} to join the SCD and improve its own payoffs by sharing the gain from removing i . Moreover, we show that the residual players cannot prevent this deviating coalition from winning a prize by considering two cases: $\{i\} > S_{k+1}$ and $\{i\} < S_{k+1}$.

If $\{i\} > S_{k+1}$, then we can swap i and S_{k+1} in an MMS partition and increase the weight of the smallest coalition in the partition (since both $S_d \cup S_{k+1} \setminus \{i\} > S_{k+1}$ and $\{i\} > S_{k+1}$). This contradicts the premise that $MMS(N, k + 1) = S_{k+1}$.

If $\{i\} < S_{k+1}$, then we show that $S_d \cup S_{k+1} > S_{k+1} = MMS(N, k + 1) > MMS(N \setminus (S_d \cup S_{k+1}), k)$: if the last ordering were false, i.e., $MMS(N \setminus (S_d \cup S_{k+1}), k) > MMS(N, k + 1)$, then we can add $S_d \cup S_{k+1}$ as a set to the MMS partition of $N \setminus (S_d \cup S_{k+1})$ and achieve a better MMS set than $MMS(N, k + 1)$, resulting in a contradiction. Since $S_d \cup S_{k+1} > MMS(N \setminus (S_d \cup S_{k+1}), k)$, $S_d \cup S_{k+1}$ can guarantee to win one of the k prizes by forming an SCD.

Having considered all cases, we have shown that there is no SCD-stable outcome in an irreducible game. \square

THEOREM 4.9 (STABILITY OF UNIFORM-PRIZE GAMES). *A k -uniform-prize WVG G has an SCD-stable outcome (i.e., a non-empty SCD-core) if and only if there are k iterative singleton winners. Moreover, an SCD-stable outcome has the unique payoff $\vec{x} = (x_1, \dots, x_n)$ such that*

$$x_i = \begin{cases} 1 & \text{if } i \text{ is one of the } k \text{ iterative singleton winners,} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. For the forward direction, suppose for a contradiction that there exists a stable outcome in a game with $\ell < k$ ISWs. Then, we can iteratively reduce the game until we are left with an irreducible game G' . By Lemma 4.8, the SCD-core of the irreducible game is empty. This means that every outcome in G' is susceptible to an SCD by some coalition C . Now, let $\omega = (\pi, \vec{x})$ be an outcome in G . If any of the ℓ ISWs in G do not receive a payoff of 1, then they can form a singleton SCD. If all of the singleton winners receive a payoff of 1, then because some coalition C has an SCD to secure some prize $p_i \in \{p_{\ell+1}, \dots, p_k\}$ in the irreducible game G' , then they also have an SCD to secure p_i in G , because the ℓ ISWs can at best only take the first ℓ prizes.

For the backward direction, suppose that there are k ISWs. Then, by definition, these ISWs can be guaranteed to win a prize on their own, and no deviation by the players who are not ISWs can secure one of these prizes. Moreover, since there are only k prizes, each ISW wins a prize and does not stand to gain by deviating. Hence, all outcomes where the k ISWs receive a payoff of 1 are stable. \square

Although this characterisation is specific to the SCD-core, it follows from Remark 2 that the MCD/ wMCD/ MCDT-core of the

game is also empty. Moreover, if the MCD/ wMCD/ MCDT-core is empty, then it cannot be the case that G contains k ISWs (because the outcome where the k ISWs each win a prize is stable to all deviation concepts).

COROLLARY 4.10. *The MCD/ wMCD/ MCDT-core of a k -uniform-prize game is non-empty if and only if it has k iterative singleton winners.*

Finally, we show that in fact, in k -uniform-prize games, the cores under all solution concepts coincide.

COROLLARY 4.11. *In a k -uniform-prize game G , $SCD\text{-core}(G) = MCD\text{-core}(G) = wMCD\text{-core}(G) = MCDT\text{-core}(G)$.*

PROOF. If G does not consist of k ISWs, then by Corollary 4.10 the core, with respect to any deviation concept, is empty. If G consists of k ISWs, then either each of the k ISWs gets payoff 1 (i.e., does not share its prize with others), or some ISWs get payoff < 1 . In the first case, we have shown that no deviation exists. In the second case, an ISW with a payoff of < 1 can deviate by forming an SCD on its own. Thus an outcome is either not stable to any deviation concept or is stable to all deviation concepts. \square

Theorem 4.9 also provides us with a polynomial time algorithm for deciding whether the core of a k -uniform-prize WVG is empty, provided that it is easy to break ties.

COROLLARY 4.12. *The problem of deciding whether a k -uniform-prize WVG has an SCD-stable outcome is in P .*

PROOF. This follows directly from Theorem 4.9: we can sort the players from largest to smallest and identify whether there are k ISWs in polynomial time by iterative reduction, which will occur at most k times. \square

5 EFFICIENCY

A natural consideration about stable outcomes is whether they are socially efficient. We consider two well-known concepts of efficiency: Pareto-optimality and utilitarian social welfare. An outcome is Pareto-optimal if there is no alternative outcome where some players can gain more without some other player losing; it is utilitarian if the total payoff for all players cannot be further increased. Observe that a utilitarian outcome is always Pareto-optimal (but not the reverse); and that, in a k -WVG, an outcome is utilitarian if and only if all prizes are claimed. Formally, an outcome $\omega = (\pi, \vec{x})$ is *Pareto-optimal* if there does not exist some $\omega' = (\pi', \vec{x}')$ such that there exists some $i \in N$ where $x'_i > x_i$, and for all $i \in N$, we have $x'_i \geq x_i$. An outcome $\omega = (\pi, \vec{x})$ is *utilitarian* if there is no outcome $\omega' = (\pi', \vec{x}')$ such that $\sum_{i \in N} x'_i > \sum_{i \in N} x_i$. A utilitarian outcome is always Pareto-optimal.

We begin with a negative result about the efficiency of SCD/ MCD-stable outcomes, followed by two positive results about wMCD/ MCDT-stable outcome.

PROPOSITION 5.1. *There are SCD/MCD-stable outcomes that are neither Pareto-optimal nor utilitarian.*

PROOF. Consider the game $G = (\vec{w}; \vec{p})$ where $\vec{w} = (0.45, 0.30, 0.25)$ and $\vec{p} = (2, 1, 1)$. The only outcome that is utilitarian, and Pareto-optimal, is $([1 \mid 2 \mid 3], (2, 1, 1))$. However, the outcome

$([2 \ 3 \mid 1], (1, 1, 1))$, where the last prize is unclaimed, is SCD/ MCD-stable as no player can gain more by deviating. \square

Next, we draw a connection between the notion of wMCD stability and Pareto optimality.

PROPOSITION 5.2. *All wMCD-stable outcomes are Pareto-optimal.*

PROOF. Suppose an outcome is wMCD-stable but not Pareto-optimal. Then all players can deviate via a wMCD to the Pareto-improved outcome, which contradicts the assumption that the outcome is wMCD-stable. \square

Finally, we find that a result analogous to the celebrated ‘Coase theorem’ [8] holds in the model we consider.

PROPOSITION 5.3. *All MCDT-stable outcomes are Pareto-optimal, even if we extend the set of feasible outcomes to allow for transfers between coalitions (i.e., replacing the requirement that for all $C \in \pi : \sum_{i \in C} x_i = v(C, \pi)$ by $\sum_{C \in \pi} v(C, \pi) = \sum_{\{j: j \leq |\pi|\}} p_j$).*

PROOF. The proof is analogous to that of Proposition 5.2. Note that the space of feasible outcomes is significantly enriched by allowing inter-coalition transfers, which also extends the Pareto-frontier. \square

6 COMPUTATIONAL COMPLEXITY

In this section, we study the computational complexity of three classes of decision problems related to the stability of a given outcome in a game, and the problem of deciding whether the core of a game is non-empty, all with respect to different deviation concepts. Henceforth, given a k -WVG G , a deviating partition $\pi_D \in \Pi(D)$ where $D \subseteq N$, and a set of target prizes $\vec{p}_t \subseteq \vec{p}$ such that $|\vec{p}_t| = |\pi_D|$, we will say that \vec{p}_t is *attainable by $\pi(D)$* if for all partitions $\pi' \in \Pi(N \setminus D)$, the i^{th} largest deviating coalition in π_D wins a prize at least as valuable as the i^{th} largest prize in \vec{p}_t . The problems are defined as follows, for $\theta \in \{SCD, MCD, wMCD, MCDT\}$:

ATTAINABLE-PRIZES:

Given: k -WVG $G = (N, \vec{w}, \vec{p}, >)$, deviating partition $\pi_D \in \Pi(D)$ where $D \subseteq N$, and target prizes $\vec{p}_t \subseteq \vec{p}$ where $|\vec{p}_t| = |\pi(D)|$.

Question: Is \vec{p}_t attainable by π_D ?

\exists - θ :

Given: k -WVG $G = (N, \vec{w}, \vec{p}, >)$, outcome $\omega = (\pi, \vec{x})$.

Question: Is ω susceptible to a θ ?

NON-EMPTY- θ -CORE:

Given: k -WVG $G = (N, \vec{w}, \vec{p}, >)$.

Question: Is the θ -core of G non-empty?

Complexity upper bounds: We begin by studying the ATTAINABLE-PRIZES problem, which we will use as an oracle in the \exists - θ problems.

LEMMA 6.1. *ATTAINABLE-PRIZES is in co-NP.*

PROOF. We can guess a partition $\pi_{N \setminus D}$ of the residual players $N \setminus D$ and verify that it prevents the deviating coalition from winning all the target prizes in polynomial time: rank all the coalitions from both the deviating partition and the residual partition, and check that the residual coalitions win one of the target prizes. This is

in NP and corresponds to the complement of ATTAINABLE-PRIZES. Therefore, ATTAINABLE-PRIZES is in co-NP. \square

With Lemma 6.1, we now derive the upper bounds on the complexity of the \exists - θ set of decision problems.

THEOREM 6.2. *For $\theta \in \{SCD, MCD, wMCD, MCDT\}$, \exists - θ is in Σ_2^P .*

PROOF. We show that checking for the existence of a θ is in NP, given access to an oracle for ATTAINABLE-PRIZES, which is in co-NP by Lemma 6.1.

We guess a deviating partition π_D and target prizes \tilde{p}_t such that the relevant incentive requirements for θ are satisfied, i.e., that if π_D attains \tilde{p}_t , then it satisfies the definition of θ . Note that for SCD, the target prize is the smallest prize which exceeds $x(D)$; for MCD, the target prizes are the smallest prizes that exceed the payoffs to each deviating coalition; for wMCD, they are the smallest prizes that equal or exceed the payoffs of each deviating coalition, with one strict inequality required; for MCDT, we only require that the sum of target prizes exceeds the total payoffs of all deviating players. We can verify that the target prizes satisfy these requirements in polynomial time.

Then, we call the ATTAINABLE-PRIZES oracle with the game G , deviating partition π_D , and target prizes \tilde{p}_t as the input. The answer is “yes” if π_D is a θ . Therefore, \exists - θ is in $\text{NP}^{\text{co-NP}} = \Sigma_2^P$. \square

Given Theorem 6.2, upper bounds on the computational complexity of deciding whether a given outcome is θ -stable and NON-EMPTY- θ -CORE naturally follow. Observing that checking for θ -stability is the complement of \exists - θ , we have the following:

COROLLARY 6.3. *For $\theta \in \{SCD, MCD, wMCD, MCDT\}$, a game G , and an outcome ω , deciding whether ω is θ -stable is in Π_2^P .*

Using an oracle to decide whether an outcome is θ -stable, we can readily derive an upper bound on the complexity of deciding whether the θ -core of a game is empty:

COROLLARY 6.4. *For $\theta \in \{SCD, MCD, wMCD, MCDT\}$, NON-EMPTY- θ -CORE is in Σ_3^P .*

While we believe the above upper bounds are tight (e.g., \exists -SCD is Σ_2^P -complete), it is an open question whether there are matching complexity lower bounds. In the remainder of this section, we provide a lower bound for the problem \exists -SCD.

Complexity lower bounds: In this section, we derive lower bounds on the computational complexity of the decision problems for the existence of beneficial deviations.

THEOREM 6.5. *\exists -SCD is NP-hard.*

PROOF. We reduce from the PARTITION problem, which is known to be NP-complete [15]. An instance of PARTITION is given by a finite set $A = \{s_1, \dots, s_n\}$ of positive integers, and the question is to determine whether there exists a subset $A' \subseteq A$ such that $\sum_{i \in A'} s_i = \sum_{i \in A \setminus A'} s_i$.

Suppose that we are given an instance $A = \{s_1, \dots, s_n\}$ of PARTITION and assume without loss of generality that the elements of A are ordered, so that $s_1 \geq s_2 \geq \dots \geq s_n$. Let $S = \frac{1}{2} \cdot \sum_{i \in A} s_i$. Now we can assume that $s_i < S$ for all $i \in \{1, \dots, n\}$ because if $s_i > S$ for some i , then the answer to PARTITION would trivially be “no”,

and if $s_i = S$ for some i , then it is easy to find the answer to PARTITION is trivially “yes” by definition of S . With this, we construct a k -WVG $G = (\tilde{w}; \tilde{p})$ with $n + 2$ players $N = \{1, 2, \dots, n + 2\}$, weights given by $\tilde{w} = \{s_1, s_2, \dots, s_n, S, S\}$, and prizes given by $\tilde{p} = (2S + \varepsilon, S, s_1, \dots, s_n)$, for some $\varepsilon < 1$.

Now, consider the outcome $\omega = (\pi, \vec{x})$, where $\pi = [n + 2 \mid n + 1 \mid 1 \mid 2 \mid \dots \mid n]$ and $\vec{x} = (s_1, \dots, s_n, S, 2S + \varepsilon)$. We assume that the tie-breaking order can be represented succinctly and is defined so that 1) $\{n + 2\} > \{n + 1\}$, 2) $\{n + 1, n + 2\} > \{1, \dots, n\}$, and 3) for any coalitions $C_1, C_2 \in \Pi(N)$ such that $w(C_1) = w(C_2)$, if player 1 is in one of these coalitions, then ties are broken in favour of the coalition containing them. We show that the answer to PARTITION is “yes” if and only if there exists an SCD from ω .

For the forward direction, suppose that A is a “yes” instance of PARTITION and let $G = (\tilde{w}; \tilde{p})$ be the k -WVG constructed as described above. Because A is a “yes” instance of PARTITION, there exists a subset $A' \subseteq A$ such that $\sum_{i \in A'} s_i = S$. Now, either A' or $A \setminus A'$ contains s_1 . Let C be the coalition of players in N with weights corresponding to either of A' or $A \setminus A'$ that contains player 1 (who has weight s_1). We will show that $C \cup \{n + 1\}$ forms an SCD from ω . To see why, note first that $w(C \cup \{n + 1\}) = 2S$ and $x(C \cup \{n + 1\}) = 2S$. Thus, the only prize that $C \cup \{n + 1\}$ will benefit from winning is the first prize, whose value is $2S + \varepsilon$. Secondly, observe that for the remaining players $R = N \setminus (C \cup \{n + 1\})$, we have $w(R) = 2S$. Since ties are broken in favour of the coalition containing player 1, i.e., coalition C , it follows that even if all of the remaining players in R join a single coalition to prevent $C \cup \{n + 1\}$ from winning first prize, ties will be broken in favour of this deviating coalition and hence, they will be able to successfully deviate to win first prize and improve their payoff. Thus, there exists an SCD from ω .

For the reverse direction, suppose that A is a “no” instance of PARTITION and consider the same construction $G = (\tilde{w}; \tilde{p})$ and outcome ω . We will show that ω is an SCD-stable outcome by considering all possible SCDs by different coalitions, which can be grouped into three categories depending on which prize they aim to win: first prize, second prize, or any other prize.

Firstly, we consider deviations that aim to secure first prize. Note that player $n + 2$ will never want to deviate from ω , because they already win the highest amount that they could possibly attain. Now, by definition of SCDs, any coalition C wanting to win first prize must satisfy $x(C) < 2S + \varepsilon$. Moreover, any coalition C that can guarantee the first prize must also satisfy $w(C) \geq 2S$. Otherwise, the remaining players could prevent them from winning the first prize by all grouping together. Now, it cannot be the case that $C = \{1, \dots, n\}$ forms an SCD, because it is assumed that $\{n + 1, n + 2\} > \{1, \dots, n\}$. Thus, if such a deviating coalition C were to exist, it must consist of player $n + 1$ and some subset $B \subseteq \{1, \dots, n\}$. However, because A is a “no” instance of PARTITION, it must be the case that either $w(B) > S$ or $w(B) < S$. If $w(B) > S$, then by construction of ω and the fact that the elements in A are all positive integers, we have $x(B \cup \{n + 1\}) \geq 2S + 1 > 2S + \varepsilon$ and hence, $B \cup \{n + 1\}$ has no incentive to deviate from ω . On the other hand, if $w(B) < S$, then $w(B \cup \{n + 1\}) < 2S$, so this coalition does not have sufficient weight to secure first prize. Thus, there is no SCD for the first prize.

Secondly, we consider deviations that aim to secure second prize. Now, such deviations must only include players from $\{1, \dots, n\}$,

as players $n + 1$ and $n + 2$ already achieve at least the value of the second prize. In this case, a similar argument holds as for the first case. If a deviating coalition C exists, then it must satisfy $x(C) < S$ and $w(C) > S$, otherwise they could not beat player $n + 1$ who wins second prize in ω . However, because ω was defined so that $w_i = x_i$ for all $i \in \{1, \dots, n\}$, it cannot be the case that both of these conditions are satisfied. Hence, there is no SCD for the second prize.

Finally, we consider the case where a subset $C \subseteq \{1, \dots, n\}$ may want to deviate to win any prize $p \in \{p_3, \dots, p_n\}$. Again, we see that any such coalition must satisfy $x(C) < p$ and $w(C) > w_j$, where w_j is the player who wins prize p in ω . However, because the considered players' weights are equal to their payoffs under ω the second condition is equivalent to $x(C) > p$ and hence, both conditions cannot be satisfied by any coalition $C \subseteq \{1, \dots, n\}$. Thus, there is no SCD for any prize. \square

Using a similar construction, we obtain the same lower bound on the complexity of \exists -MCD and \exists -wMCD. We omit the proof here as the line of reasoning is similar to that of Theorem 6.5.

THEOREM 6.6. \exists -MCD and \exists -wMCD are NP-hard.

7 RELATED WORK

The related models of standard WVGs and PFGs, of which k -WVGs are a subset, have been the subject of extensive study. We discuss some of the related literature here.

Complexity of standard WVGs: The stability of standard WVGs has been well-studied (see, for example, [6]). It is shown in [9] that deciding if the core of a WVG is non-empty is NP-hard and in Σ_2^P and the related problem of deciding whether an outcome is in the core is co-NP-complete. In [12], it was shown that the upper bound can be lowered to Δ_2^P .

Characteristic function games and partition function games: As we noted, the standard WVG is a class of characteristic function games (see for example [9]), while k -WVGs are a class of PFGs (see for example [17, 23–25]), where each partition of players can invoke a different characteristic function for the coalitions within that partition. Such games can also be represented using embedded MC-nets [21], which have the benefit of being fully expressive and exponentially more concise than the conventional PFG representation in some cases. Our model is always exponentially more concise than the general representation, but focuses on a natural and intuitive subset of all PFGs.

Games with multiple prizes and ranking: One class of characteristic function games that involves multiple prizes are known as *Threshold Task Games* (TTGs) [11]. In TTGs, there are several tasks, each with an associated threshold and value, such that the value of a coalition is given by the highest-valued task whose associated threshold is no greater than the coalition's weight. As with standard WVGs, there are no externalities to coalition formation in TTGs, which are an essential feature of k -WVGs.

The idea of having the outcomes of a game depend on a *ranking* of its participants, be they players or coalitions, was also proposed in a *non-cooperative* setting by Brandt et al. in [4]. However, in notable distinction to our work, the outcomes of their so-called

Ranking Games are rankings over players rather than over coalitions of players. Moreover, they focused on Nash equilibrium, correlated equilibrium, and the price of anarchy, rather than on more cooperative stability concepts like the core and its variants.

Assumptions about the residual players: Many different concepts of stability exist in PFGs. One dimension in which they differ is how the deviating players expect the residual players to respond: the α -core [2] assumes that the residual players would act against the deviating coalition, such that a deviation is only feasible if the deviating coalition is better off under all partitions of the residual players; the ω -core [22] assumes that the residual players would act favourably toward the deviating coalition, such that a deviation is feasible if the deviating coalition is better off under some partition of the residual players; the s -core [7] assumes that the residual players would form singleton coalitions; the m -core [1, 13, 19, 20] assumes that the residual players form a single coalition; and the δ -core [14] assumes that the residual players do not react and remain in their respective coalitions. Our concept of the SCD-core for the k -WVG coincides with the α -core of the more general PFGs.

More nuanced assumptions about the residual players can also be made where various degrees of incentive compatibility are involved. Interested readers should see [17] for a survey of stability concepts.

Deviations involving multiple coalitions: Another dimension in which deviation concepts differ in PFGs is whether the deviating players form a single coalition or partition into multiple coalitions. In [3, 18], deviating players can form a partition if doing so increases the total value of all deviating coalitions. This corresponds to the concept of MCDT in this paper, where we implicitly allow transfers *between* coalitions to satisfy the incentive compatibility requirements of the deviating coalitions.

Efficiency It has been noted in [17] that, in partition form games, an outcome that is stable to some deviation concept by a single coalition may not be Pareto-efficient. The simple intuition here is that the deviating coalition may gain a greater collective value if they instead form a partition of multiple deviating coalitions. [16] removes this inefficiency by considering outcomes that are stable to deviations involving multiple coalitions. In Section 5 we showed that this is indeed true for k -WVGs. However, we also showed that in k -WVGs, wMCD-stability is sufficient for ensuring Pareto-efficiency without resorting to inter-coalition transfers.

8 CONCLUSIONS

k -WVGs provide a model for studying coalition formation where coalitions compete to win prizes based on their relative strengths. We began exploring some of its interesting properties, including stability concepts, the emptiness of the core, the computational complexity of stability, and the efficiency of stable outcomes. Further directions of research may include: finding tight complexity bounds for solution concepts, further characterise the core for games with all players and all prizes, exploring fairness notions, adopting alternative assumptions about the residual players in a deviation (e.g., dispensing with the pessimistic outlook), and whether, given a set of players, we can design prizes in order to achieve a particular set of stable outcomes.

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