Mechanism Design for Improving Accessibility to Public Facility

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ABSTRACT

We consider a variant of the facility location problems where agents are located on a real line and the facility is fixed at a designated location to serve the agents. As the facility cannot be relocated due to various constraints (e.g., construction costs and regulatory requirements), the social planner considers the structural modification problem of adding a short-cut edge to the real line for improving the accessibility or costs of the agents to the facility, where the cost of an agent is measured by their shortest distance to the facility. For a mechanism design version of the structural modification problem where the agents are assumed to have private locations, we propose several strategy-proof mechanisms to elicit truthful locations from the agents and add a short-cut edge to (approximately) minimize the total cost or maximum cost of agents. We derive the upper bounds of these mechanisms and provide lower bounds on the approximation ratios for both objectives.

KEYWORDS

facility location, mechanism design, approximation ratio

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1 INTRODUCTION

In recent years, facility location problems (FLPs) have been well-studied in the context of approximate mechanism design without money [2, 6, 22]. In the mechanism design version of the FLPs, a social planner seeks to locate a facility to serve a set of agents located within a metric space (e.g., on a real line) such that the located facility minimizes a given cost objective that measures the distance of the agents to the facility. The locations of the agents in the metric space are private, and each agent has the potential to misreport their own location to make the facility closer to them. Therefore, the social planner's goal is to design a strategy-proof mechanism that elicits truthful location information from the agents and locates a facility that (approximately) optimizes a given cost objective. The FLPs can be used to model realistic scenarios including determining the location of a physical facility (e.g., a public library, school, or park) to serve a population [6].

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Our Problem and Approach. In many situations, the location of the facility (such as library and park) has already been predetermined previously even though they can become undesirable or sub-optimal for the agents. For example, the undesirability and sub-optimality can be the results of the change in agent compositions, where the population in the targeted domain (e.g., regions and areas) are different from the past (e.g., moving of the agents to new locations), or the change of the social planner's objective, where the social objective is now defined based on a new objective that is different from the past. One direct approach to addressing this sub-optimality is to "ignore" the previous facility and locate a new facility subject to the new objective. However, in many situations, the planner simply cannot discard the previous facility due to internal and external constraints. For instance, when a facility such as a school or a library has been located, one cannot simply construct a new facility to improve accessibility without considering the space, time, legal regulations, and costs required for the new facility. Thus, we need a new idea that can help the social planner to improve the accessibility for the agents when creating a new facility is not entirely possible.

Recognizing this challenge, existing optimization literature on facility location (e.g., see [3, 4, 25]) has proposed to structurally modify the metric space (i.e., discrete networks in their settings) to minimize the costs of the agents to the facility. The main interpretation of the structural modification approach is that the social planner can build new edges (e.g., roadways or bridges) or provide shuttle services between two points in existing metric space to reduce the distances of the agents to the facility.

Following the existing approach to improve agent accessibility to the facility, in this paper, we consider the FLPs from a structural modification perspective by modifying the metric space (i.e., a real line) to minimize the costs of agents to the facility when the facility's location has been fixed. In particular, we focus on a simple, yet challenging modification methodology, of adding a costless short-cut edge (e.g., a shuttle service) to the metric space.

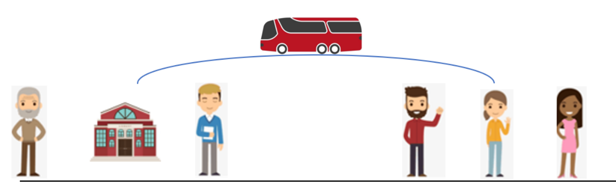

Figure 1: An example where a shuttle takes the agents from one position to the facility directly.

Such FLPs can model a wide range of settings in which social planners provide free shuttle services to the agents. In our context, the shuttle service takes the agents directly from one central location a to a point b near the facility without any stop (see Figure 1). For instance, a shuttle that takes students on campus to their car parking lot (for commuting students), academic building, or shopping mall (for students living on campus). A shuttle may also take residents in a community to their local library. By using the shuttle, each agent does not incur any traveling costs. Since the shuttle is usually free, we assume that the short-cut edge has zero cost. Given the facility's location, we aim to design strategy-proof mechanisms to elicit truthful location information from the agents and add a short-cut edge to the space that (approximately) optimizes given objectives.

To our best knowledge, no existing work studies the design of mechanisms for our proposed setting. As discussed earlier, several works study this variant of FLPs as optimization problems [3, 4, 25] on discrete networks.

Our Challenges and Contribution. Our main conceptual contribution is the investigation of the FLPs from a structural modification perspective within the approximate mechanism without money paradigm [22]. More specifically, we focus on the most fundamental setting of FLPs where the metric space is a real line. Given this setting, we are interested in adding a short-cut edge (e.g., shuttle) to the real line to improve agents' costs. The line space is the most studied one in the literature on FLPs, as initiated in the original work of approximate mechanism design without money [22] since they can model various natural structures such as the street.

In our variant of FLPs of adding a short-cut edge, there are n agents and a facility located on a point in the line. Taking the agent locations as input, a mechanism selects two points a, b in the line and forms a short-cut edge (a, b) with the cost of 0 to the agents. The (new) cost of an agent is simply the length of the shortest path from the facility. Intuitively, the zero-cost short-cut edge models some forms of services or resources provided by the planner that decreases the travel cost of the agents. For instance, the short-cut edge can be viewed as a pair of pickup or drop-off points for a facility shuttle, which is free for all agents. The agents do not incur any cost for using the service. In this paper, we focus on the maximum cost objective, which aims to minimize the maximum costs among all agents, and the social cost objective, which aims to minimize the total cost of all agents.

In many existing mechanism design variants of FLPs, the agents have single-peaked preferences [5], where [19] provide a characterization of strategy-proof mechanisms. As a result, existing works can leverage the characterization to design strategy-proof mechanisms with good approximation ratios. Unfortunately, agents in our variant of FLPs do not have single-peaked preferences.Existing mechanisms for FLPs no longer apply directly, and it is not clear what mechanisms can be strategy-proof and obtain good approximation ratios. Thus, designing such mechanisms for our variant of FLPs can be more difficult and challenging. For example, the mechanisms that connect the facility with the median agent or the farthest agent either are not strategy-proof or do not provide a good approximation ratio. However, we are able to obtain results (both the new mechanisms and lower bounds) that provide a first attempt to address the new variant of FLPs. The results are summarized in Table 1. Omitted proofs are in the Appendix.

Table 1: A summary of our results.

Objective	Deterministic	Randomized
Maximum cost	UB: 3 LB: 1.5	UB: 2.75 LB: 1.5
Social cost	UB: n LB: 1.5	UB: 6 LB: 1.02

Section 3 is concerned with the maximum cost objective. We prove that a TwoExtreme mechanism that adds an edge connecting the two extreme (leftmost and rightmost) agents is strategy-proof and 3-approximation. We introduce a new randomized mechanism (see Mechanism 2, which is the main contribution of this section) to improve the approximation ratio to 2.75. This mechanism first designates three points based on leftmost and rightmost agents and then connects each of them with the facility with probabilities $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{2}$, respectively. Further, a lower bound of 1.5 is provided for all randomized strategy-proof mechanisms.

Section 4 is concerned with the social cost objective. While the TwoExtreme mechanism is n-approximate, we propose a new randomized 6-approximation mechanism (see Mechanism 3), which connects the facility with each agent's location with a probability proportional to the distance from the facility. This mechanism uses a similar idea to the proportional mechanism for the FLPs with two facilities [16]. We present lower bounds 1.5 and 1.02 for deterministic and randomized strategy-proof mechanisms, respectively.

Quality of Our Results. In order to highlight the quality of our lower and upper bounds, we compare our results to the mechanism design results of FLPs with two facilities (or 2-FLPs). The reason is that, in our setting, there is one existing facility and the edge added can be viewed as another (extended) facility. Note that the mechanisms for FLPs cannot work for our setting directly because the agents' preferences are no longer single-peaked. Table 2 summarizes the best-known results for 2-FLPs after years of research attempts.

Table 2: A summary of the best-known results for FLPs with two facilities (2-FLPs) [6].

Objective	Deterministic	Randomized
Maximum cost	UB: 2 LB: 2	UB: 5/3 LB: 1.5
Social cost	UB: $n-2$ LB: $n-2$	UB: 4 LB: 1.045

While our results are not tight for deterministic mechanisms, for maximum cost, we obtain a similar range of upper and lower bounds as 2-FLPs. For the social cost, we hypothesize that the lower bound can be much higher (e.g., $\Omega(n)$) given the $(n-2)$ results from 2-FLPs. We note that the techniques used for proving the lower bound $n-2$ in 2-FLPs [14] cannot directly work for our problem, because in our problem a solution consists of two points, whose possible locations are more difficult to be constrained by the approximation ratio and strategyproofness.

For randomized mechanisms, the best-known results for FLPs and our problem are not tight. In fact, for the maximum cost, we obtain the same lower bound as the 2-FLPs, and our upper bound is not too far from 2-FLPs. For the social cost, our lower bound is similar to those of 2-FLPs. Our upper bound is within a factor of 1.5 of the 2-FLPs.

Thus, our lower and upper bounds (established non-trivially) are a reasonable initial results for a new problem.

Related Work. Our work is grounded on a string of research for facility location and network modification problems.

Facility location. The mechanism design problem for FLPs is a problem where the agents report their private locations in some metric space, and the social planner determines the facility's location based on these reports and the planner's objective function. There are many works that focus on designing strategy-proof mechanisms. For example, Schummer and Vohra [23] characterize deterministic strategy-proof (SP) mechanisms on the line and tree, Dokow *et al.* [10] give a full characterization of SP mechanisms on discrete lines and cycles, and Alon *et al.* [1] study the maximum cost objective on both continuous and discrete graphs. More than a decade ago, Procaccia and Tennenholtz [22] put forward the agenda of approximate mechanism design without money, which advocates the study of strategy-proof mechanisms for various optimization problems through the lens of the approximation ratio. They provide tight bounds on the approximation ratio of SP mechanisms for the single-facility problem. Later, [16, 17] improve the bound for two-facility problems. After that, many variants centered around this agenda are extensively studied, for example, preferences over facilities [12, 24], distance constraint [7, 8], and different cost functions [11, 13]. See [6] for a survey. All these works consider how to locate the facilities to optimize cost objectives. We focus on how to modify the structure to minimize agents' costs when the facility's location is fixed.

Edge addition on networks. In previous literature, there are edge addition optimization problems on discrete networks [9, 18, 20, 21] related to our work. In these problems, the planner aims to add shortcut edges to a network to minimize some graph parameters [4] (e.g., the diameter and average distances) or find the minimum number of edges to be added to the graph such that the resulting diameter is no greater than a given number [15]. The works of [3, 25] design algorithms to minimize the distance of agents to the facility for trees and general networks. All optimization studies above focus on discrete networks without considerations of strategic agents.

2 PRELIMINARIES

Let $N = \{1, 2, \ldots, n\}$ be the set of agents. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the location profile of agents, where $x_i \in \mathbb{R}$ is the location of agent $i \in N$. The facility has a publicly known location on the line. Without loss of generality, the facility's location is located at point 0. We want to select two points a, b on the line and connect them via a short-cut edge of length 0. By using the short-cut edge, each agent does not incur any traveling costs (see Section 1 for examples).

A (deterministic) mechanism is a function $f : \mathbb{R}^n \to \mathbb{R}^2$ that maps a (reported) location profile $\mathbf{x}$ to two points for a new edge. Given a solution $f(\mathbf{x}) = (a, b)$, the *cost* of agent $i \in N$ is

$$cost((a, b), x_i) = \min\{|x_i|, |x_i - b| + |a|, |x_i - a| + |b|\},$$

that is, the minimum distance to the facility in the modified space. A randomized mechanism is a function f from $\mathbb{R}^n$ to a probability distribution over $\mathbb{R}^2$. When $f(\mathbf{x})$ is a probability distribution $\mathbf{P}$, the cost of agent $i \in N$ is the expected distance to the facility, i.e., $cost(\mathbf{P}, x_i) = \mathbb{E}_{(a,b)\sim\mathbf{P}} cost((a, b), x_i)$.

Example 2.1. Consider a location profile $\mathbf{x} = (-2, 3)$ with 2 agents. The cost of agent 2 is equal to 3. If the planner adds a new edge $(-1, 2)$ to the line, then the cost of agent 2 decreases to $|x_2 - 2| + |-1| = 2$.

A mechanism f is *strategy-proof*, if no agent can benefit from misreporting a false location, regardless of the strategies of other agents. That is, for all $\mathbf{x} \in \mathbb{R}^n$, $i \in N$ and all $x_i' \in \mathbb{R}$, we have $cost(f(\mathbf{x}), x_i) \le cost(f(x_i', \mathbf{x}_{-i}), x_i)$, where $\mathbf{x}_{-i}$ is the profile of the locations of all agents in $N\backslash\{i\}$.

Objectives. In this paper, we shall be interested in strategy-proof mechanisms that approximately optimize one of two objective functions: minimizing the social cost, or minimizing the maximum cost. The *social cost* of a solution $s = (a, b)$ of edges with respect to location profile $\mathbf{x}$ is $sc(s, \mathbf{x}) = \sum_{i\in N} cost(s, x_i)$; the social cost of a distribution $\mathbf{P}$ w.r.t. $\mathbf{x}$ is $sc(\mathbf{P}, \mathbf{x}) = \mathbb{E}_{s\sim\mathbf{P}}\left[sc(s, \mathbf{x})\right]$. The *maximum cost* of a solution s (resp. a distribution $\mathbf{P}$) with respect to location profile $\mathbf{x}$ is $mc(s, \mathbf{x}) = \max_{i\in N} cost(s, x_i)$ (resp. $mc(\mathbf{P}, \mathbf{x}) = \mathbb{E}_{s\sim\mathbf{P}}\left[mc(s, \mathbf{x})\right]$).

We say a mechanism is α*-approximation* (or equivalently, has *approximation ratio* α) for $\alpha \ge 1$, if for all instances, the objective value induced by the mechanism is no greater than α times the optimal value.

Given location profile $\mathbf{x}$, let $x_l = \min_{i\in N} x_i$ and $x_r = \max_{i\in N} x_i$ be the locations of the leftmost and rightmost agents, respectively. For ease of notations, we assume there is always a dummy agent located at 0 (unless the location profile is specified), which indicates that $x_l \le 0, x_r \ge 0$.

It is easily observed that any edge e that is added to the line in order to minimize the distance from the facility at 0 is only used in one direction by all shortest paths from 0 that use this edge e, and thus there must be an optimal solution that adds an edge with an endpoint 0. This key proposition will be used throughout this paper to compute the optimum.

Proposition 2.2. *For both objectives, there exists an optimal solution* $(0, y)$ *for some point* $y \in \mathbb{R}$.

Proof. First, we show that no new edge can be used by agents from both sides. Consider any solution (a, b) with $a \le b$ and any two agents $i, j \in N$ with $x_i < 0, x_j > 0$. If both i, j improve by using this solution, then it must be $|x_i - a| + |b| < |x_i|$ and $|x_j - b| + |a| < x_j$; however, this is impossible. So either i or j or none of them can improve by using (a, b).

Second, let (a^*, b^*) with $a^* \le b^*$ be an optimal solution, and denote by S the set of agents who gain by using (a^*, b^*). Assume w.l.o.g. that it is used by some agents on the right side, i.e., for all $i \in S$, $x_i > 0$. Then clearly $b^* > 0$. The cost of each agent $i \in S$ is $|x_i - b^*| + |a^*|$. However, using another solution $(0, b^*)$, the cost of every agent $i \in S$ decreases to $|x_i - b^*|$. So $(0, b^*)$ is also an optimal solution, as desired. ◻

3 MINIMIZING THE MAXIMUM COST

We study the maximum cost objective in this section. Given location profile $\mathbf{x}$, if $|x_l| \le x_r$, define $l(\mathbf{x}) = \min\{x \in \mathbf{x} \mid x > \frac{x_r}{3}\}$, and $b(\mathbf{x}) = \max\{x \in \mathbf{x} \mid 0 \le x \le \frac{x_r}{3}\}$. Intuitively, when there is an edge connecting point $\frac{2x_r}{3}$ with the facility, all agents located in $l(\mathbf{x})$ would like to use this edge, while the agents located in $b(\mathbf{x})$ go to the facility directly. If $|x_l| > x_r$, symmetrically, define $l(\mathbf{x}) = \max\{x \in \mathbf{x} \mid x < \frac{x_l}{3}\}$, and $b(\mathbf{x}) = \min\{x \in \mathbf{x} \mid \frac{x_l}{3} \le x \le 0\}$. We write $l(\mathbf{x}), b(\mathbf{x})$ as l, b, when no confusion arises.

Proposition 3.1. *Given location profile $\mathbf{x}$, if $|x_l| \le x_r$, then edge $(0, \frac{l+x_r}{2})$ is an optimal solution, and the optimal maximum cost is $OPT(\mathbf{x}) = \max\{|x_l|, b, \frac{x_r-l}{2}\}$. Symmetrically, if $|x_l| > x_r$, the optimal maximum cost is $OPT(\mathbf{x}) = \max\{x_r, |b|, \frac{x_l-l}{2}\}$.*

Proof. Consider $|x_l| \le x_r$. Recall from Proposition 2.2 that there is an optimal solution that connects some point with the source 0, say $(0, y^*)$. Noting that any edge can facilitate the agents in only one direction, at least one of the two extreme agents cannot gain from the new edge, and we have $OPT(\mathbf{x}) \ge \min\{|x_l|, x_r\} = |x_l|$. If $y^* < \frac{2x_r}{3}$, then the cost of the agent on x_r is at least $x_r - y^* > \frac{x_r}{3} \ge b$. If $y^* \ge \frac{2x_r}{3}$, then the cost of the agent on b is b. So we have $OPT(\mathbf{x}) \ge b$. Now it remains to prove $OPT(\mathbf{x}) \ge \frac{x_r-l}{2}$. Considering the agents on l and x_r, the best possible case for balancing these two agents is when y^* is the midpoint of l and x_r, and thus $mc((0, y^*), \mathbf{x}) \ge \frac{x_r-l}{2}$. Combining the above, we have $OPT(\mathbf{x}) \ge \max\{|x_l|, b, \frac{x_r-l}{2}\}$. On the other hand, it is easy to see that $(0, \frac{l+x_r}{2})$ achieves a maximum cost of at most $\max\{|x_l|, b, \frac{x_r-l}{2}\}$, which indicates the optimality. □

Note that optimal solution is not strategy-proof. Consider an example with two agents located at $\mathbf{x} = (4, 6)$. The optimal solution is connecting $(0, 5)$, and the agent located at 6 incurs a cost of 1. However, if this agent misreports her location as 8, the location profile becomes $\mathbf{x}' = (4, 8)$, and the optimal solution for $\mathbf{x}'$ is $(0, 6)$. Thus, the agent located at 6 can decrease her cost to zero after misreporting.

3.1 Deterministic Mechanisms

The following mechanism deterministically connects the two extreme locations x_l and x_r of agents. Due to the existence of a dummy agent on 0, the left (right) endpoint of the new edge is either 0 or to the left (right) of 0.

Mechanism 1. *Given agents' location profile $\mathbf{x} = (x_1, \ldots, x_n)$, add new edge $f(\mathbf{x}) = (x_l, x_r)$.*

Lemma 3.2. *Mechanism 1 is strategy-proof.*

Proof. Let $\mathbf{x}' = (x'_i, \mathbf{x}_{-i})$ be the location profile when some agent i misreports x'_i, and let x'_l, x'_r be the two extreme locations with respect to $\mathbf{x}'$. By symmetry, it suffices to prove that any agent i to the right of 0 (i.e., $x_i > 0$) will not lie. If i is not an extreme agent, i.e., $0 < x_i < x_r$, then it must have $x'_l \le x_l$ and $x_r \le x'_r$. After misreporting, the cost of agent i becomes $cost(f(\mathbf{x}'), x_i) = \min\{x_i, x'_r - x_i + |x'_l|\} \ge \min\{x_i, x_r - x_i + |x_l|\} = cost(f(\mathbf{x}), x_i)$, implying that i cannot gain by misreporting. If i is an extreme agent with $x_i = x_r$, the cost after misreporting becomes $cost(f(\mathbf{x}'), x_i) = \min\{x_r, |x'_r - x_r| + |x'_l|\} \ge \min\{x_r, |x_l|\} = cost(f(\mathbf{x}), x_i)$, and thus there is no incentive to lie. □

We remark that Mechanism 1 indeed satisfies a stronger concept called group strategy-proofness [22], which means that for any location profile $\mathbf{x}$ and any coalition $S \subseteq N$, there is no joint deviation $\mathbf{x}_S$ of the agents in S such that all the agents in S gain. Now we prove an approximation ratio of 3.

Theorem 3.3. *For the maximum cost objective, Mechanism 1 is deterministic, strategy-proof and 3-approximation.*

Proof. The strategy-proofness is given by Lemma 3.2. For the approximation ratio, we only consider the case $|x_l| \le x_r$, as the other case $|x_l| > x_r$ is symmetric. By Proposition 3.1, the optimal maximum cost $OPT(\mathbf{x})$ is $\max\{|x_l|, b, \frac{x_r-l}{2}\}$. Clearly, the cost of each agent i with $x_i < \frac{x_r}{3}$ is at most $|x_i| \le \max\{|x_l|, b\} \le OPT(\mathbf{x})$. It remains to consider each agent i with $x_i \ge l \ge \frac{x_r}{3}$. The distance of agent i to x_r is $x_r - x_i$, and thus the cost of agent i is at most

$$\begin{aligned} x_r - x_i + |x_l| &\le x_r - l + |x_l| \\ &\le 2 \cdot \frac{x_r - l}{2} + |x_l| \\ &\le 3 \cdot OPT(\mathbf{x}), \end{aligned} \tag{1}$$

which gives an approximation ratio of 3. □

The following example shows that the approximation ratio 3 of Mechanism 1 can be attained. Recall that the dummy is not needed when a location profile is specified.

Example 3.4. Consider a 3-agent instance with location profile $\mathbf{x} = (-1, 8, 10)$. The optimal maximum cost is $OPT(\mathbf{x}) = 1$, attained by solution $(0, 9)$. Mechanism 1 adds edge $(-1, 10)$, and the cost of agent 2 is 3. Thus, the ratio $\frac{3}{OPT(\mathbf{x})} = 3$.

3.2 Randomized Mechanisms

In the following, we consider randomized mechanisms.

Mechanism 2. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the location profile of agents. If $|x_l| \le x_r$, add an edge $f(\mathbf{x}) = (x_l, y)$, where y is a random point defined as follows.*

- *Case 1. $l(\mathbf{x}) \ge \frac{2x_r}{3}$. Let $c(\mathbf{x}) = \max\{|x_l|, \min\{l(\mathbf{x}), x_r - b(\mathbf{x})\}\}$, and*

$$y = \begin{cases} c(\mathbf{x}) \text{ with probability (w.p.) } \frac{1}{4} \\ x_r \text{ w.p. } \frac{1}{2} \\ \frac{c(\mathbf{x})+x_r}{2} \text{ w.p. } \frac{1}{4} \end{cases}$$

- *Case 2. $l(\mathbf{x}) < \frac{2x_r}{3}$. Let $d(\mathbf{x}) = \max\{|x_l|, \frac{2x_r}{3}\}$, and*

$$y = \begin{cases} d(\mathbf{x}) \text{ w.p. } \frac{1}{4} \\ x_r \text{ w.p. } \frac{1}{2} \\ \frac{d(\mathbf{x})+x_r}{2} \text{ w.p. } \frac{1}{4} \end{cases}$$

If $|x_l| > x_r$, $f(\mathbf{x}) = (y, x_r)$ is defined symmetrically.

Intuitively, this mechanism designates a point (i.e., $c(\mathbf{x})$ or $d(\mathbf{x})$) and connects the facility with this point, x_r and their midpoint with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively.

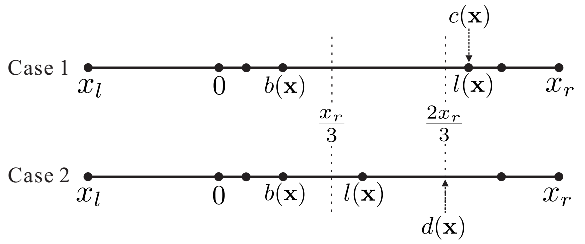

Figure 2: Illustration of Mechanism 2.

Figure 2 gives an illustration of the notations in Mechanism 2. It is worth noting that we have $c(\mathbf{x}) \geq d(\mathbf{x}) \geq \frac{2x_r}{3}$ whenever $|x_l| \leq x_r$ and $l(\mathbf{x}) \geq \frac{2x_r}{3}$. We write $c(\mathbf{x}), d(\mathbf{x})$ as c, d, when no confusion arises. Before the analysis, we give an example to show how the mechanism works and how bad the performance can be.

Example 3.5. Consider an instance with agents' location profile $\mathbf{x} = (\frac{-L}{2}, 0, 2L, 3L)$ for some large number $L > 0$. Then $x_l = \frac{-L}{2}, x_r = 3L, b(\mathbf{x}) = 0, c(\mathbf{x}) = l(\mathbf{x}) = 2L$. Since $l(\mathbf{x}) \geq \frac{2x_r}{3}$, the solution returned by this mechanism is $(\frac{-L}{2}, 2L), (\frac{-L}{2}, 3L), (\frac{-L}{2}, 2.5L)$ with probability $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. The maximum cost of the mechanism in expectation is $\frac{1}{4} \cdot (L + |\frac{-L}{2}|) + \frac{1}{2} \cdot (L + |\frac{-L}{2}|) + \frac{1}{4} \cdot (\frac{L}{2} + |\frac{-L}{2}|) = \frac{11L}{8}$. The optimal solution is $(0, 2.5L)$, and the optimal maximum cost is $0.5L$. Hence, the maximum cost of the mechanism for this instance is $\frac{11}{4}$ times of the optimum.

We show the strategy-proofness of Mechanism 2.

Lemma 3.6. *Mechanism 2 is strategy-proof.*

Proof. Let $\mathbf{x}$ be a location profile with $|x_l| \leq x_r$. We first consider any agent i to the left of 0, i.e., $x_i < 0$. When telling the truth, the cost of agent i is at most $|x_i|$. Suppose she misreports a location x_i', and the location profile becomes $\mathbf{x}' = (x_i', \mathbf{x}_{-i})$. If $x_i' < x_l$ and $|x_i'| \leq x_r$, the solution does not change. If $x_i' < x_l$ and $|x_i'| > x_r$, then a new edge connects some point to the left of 0 with x_r, which cannot decrease her cost. If $x_i' \geq 0$, then the solution is $f(\mathbf{x}') = (x_l', y')$ for some random point $y' \geq |x_l'|$, and the cost is still $|x_i|$. So agent i cannot gain by misreporting.

Next, we consider agent i to the right of 0, i.e., $x_i > 0$. We claim that agent i will not report a location x_i' such that $x_l' \neq x_l$, which enables us to focus on the case $x_l = x_l'$.

Claim 1. *If agent i with $x_i > 0$ misreports a location $x_i' < 0$ such that $x_l' \neq x_l$, she cannot decrease the cost.*

Proof. When $x_l' \neq x_l$, there are two cases: $|x_l'| > x_r'$, and $x_r' \geq |x_l'| > |x_l|$. Note that the cost when telling the truth is at most x_i. First, we consider the case when $|x_l'| > x_r'$. For agent i with $0 < x_i < x_r$, when she misreports x_i' such that $|x_l'| > x_r' = x_r$, the cost is at least $\min\{x_i, |x_r - x_i| + x_r\} = x_i$, which indicates that she will not lie. For agent i with $x_i = x_r$, when she misreports x_i' such that $|x_l'| > x_r'$, the cost is at least $\min\{x_r, |x_r - x_r'| + x_r'\} = x_r = x_i$. So she has no incentive to lie.

Second, we consider the case when $x_r' \geq |x_l'| > |x_l|$. For agent i with $x_i > 0$, by changing x_l to x_l', the corresponding loss in the left side is $|x_l' - x_l|$; however, the benefit in the right side is at most $|x_l' - x_l|$, implying that she cannot gain. Claim 1 is proved.

Now we focus on the case $x_l = x_l'$. For agent i with $0 < x_i \leq \frac{x_r}{3}$, note that she always goes to the source directly in every case. The cost is x_i, and does not change whenever misreporting.

For agent i with $\frac{x_r}{3} < x_i < \frac{2x_r}{3}$, such agents exist only in Case 2 of Mechanism 2. If she misreports x_i' such that the location profile $\mathbf{x}'$ is still in Case 2, because $x_i < d \leq d'$, she cannot improve. If she misreports x_i' such that $\mathbf{x}'$ falls in Case 1, because $x_r \leq x_r'$, $|x_l'| = |x_l|$ and $x_i < d \leq c \leq c'$, she cannot improve.

For agent i with $\frac{2x_r}{3} \leq x_i \leq x_r$, we discuss the two cases respectively.

Case 1. $l(\mathbf{x}) \geq \frac{2x_r}{3}$. If none of $x_r, l(\mathbf{x}), b(\mathbf{x})$ changes after misreporting, the cost does not change as well. If b changes and x_r does not change, it must be still in Case 1, and we discuss two subcases as follows.

- Case a. $l(\mathbf{x}) \leq x_r - b$. It implies $c = \max\{|x_l|, l\}$. When $x_i \neq l$, it must be either $c = c' = |x_l|$, or $c' \leq c = l < x_i$: in the former case, the solution does not change; in the latter case, by the probability distribution, agent i cannot improve. When $x_i = l$, if $c = |x_l|$, because $x_i \leq x - |x_l| \leq c'$, agent i cannot improve; if $c = l = x_i$, by the probability distribution, the cost of agent i will not decrease no matter how c moves.
- Case b. $x_r - b(\mathbf{x}) < l(\mathbf{x})$. It implies that $c = \max\{|x_l|, x_r - b\}$. By $b' \geq b$, it must be $c' \leq c$. Then it is easy to see that, whenever, $c = |x_l| < x_i$ or $l(\mathbf{x}) \leq x_i \leq x = |x_l|$ or $c = x_r - b < l(\mathbf{x}) \leq x_i$, the cost cannot decrease.

If x_r, b do not change and l decreases, we discuss two subcases $l(\mathbf{x}') \geq \frac{2x_r}{3}$ and $l(\mathbf{x}') < \frac{2x_r}{3}$.

- Case c. $l(\mathbf{x}') \geq \frac{2x_r}{3}$. It must be $c' \leq c$. The only possible way for improvement is $l(\mathbf{x}) \leq x_i <= |x_l|$; however, in this case, $c = c'$, and thus the cost is the same.
- Case d. $l(\mathbf{x}') < \frac{2x_r}{3}$. Clearly, we have $d' = d \leq c$. The only possible way for improvement is $l(\mathbf{x}) \leq x_i < c = |x_l|$; however, in this case, $c = |x_l| = d$, and thus the cost is the same.

If x_r, b do not change and l increases (implying $l(\mathbf{x}) = x_i$), we also discuss two subcases $l(\mathbf{x}') \geq \frac{2x_r}{3}$ and $l(\mathbf{x}') < \frac{2x_r}{3}$.

- Case e. $l(\mathbf{x}') \geq \frac{2x_r}{3}$. It must be $c' \leq c$. If $x_r - b \leq l$, then $c' = c$, and thus the cost is the same. If $l < x_r - b$, then $x_i = l \leq c \leq c'$, and agent i cannot improve.
- Case f. $l(\mathbf{x}') < \frac{2x_r}{3}$. Clearly, we have $d' = d \leq c$. The only possible way for improvement is $x_i = l < c = |x_l|$; however, in this case, $c = |x_l| = d'$, and thus the cost is the same.

Now we discuss when x_r changes. If $x_r' < x_r$, then it must be $x_i = x_r$, and is easy to see that agent i cannot decrease the cost, because the random point y moves to the left in expectation. If $x_r < x_r'$, let $\Delta = x_r' - x_r$ be the difference. When $f(\mathbf{x}')$ comes from Case 1, we have $c' - c \leq \Delta$ and $\frac{x_r' + c'}{2} - \frac{x_r + c}{2} \leq \Delta$. By the probability distribution and $x_i \leq x_r$, the expected cost of agent i will never decrease. Similarly, when $f(\mathbf{x}')$ comes from Case 2, we have $|d' - c| \leq \Delta$ and $\frac{x_r' + d'}{2} - \frac{x_r + c}{2} \leq \Delta$, and also the expected cost of agent i will never decrease.

Case 2. $l(\mathbf{x}) < \frac{2x_r}{3}$. If x_r does not change (i.e., $x'_r = x_r$), then $\mathbf{x}'$ is still in Case 2, and by $d' = d$, the solution also does not change. So it suffices to consider two subcases $x'_r < x_r$ and $x_r < x'_r$. Let $f(\mathbf{x}) = (x_l, y)$ and $f(\mathbf{x}') = (x_l, y')$.

- Case g. $x'_r < x_r$. It implies $x_i = x_r$. If $\mathbf{x}'$ is still in Case 2, we have $d' < d$, and the expectation of y' is smaller than that of y. So the cost of agent i will increase. If $\mathbf{x}'$ is in Case 1, then there must be an agent j with $x_j = l(\mathbf{x}') \leq \frac{2x_r}{3}$. It follows that $c' \leq d$, and the expectation of y' is smaller than that of y. Therefore, agent i has no incentive to lie.
- Case h. $x_r < x'_r$. Let $\Delta = x'_r - x_r$ be the difference. If $\mathbf{x}'$ is still in Case 2, we have $|d' - d| \leq \Delta$ and $|\frac{x'_r+d'}{2} - \frac{x_r+d}{2}| \leq \Delta$. By the probability distribution and $x_i \leq x_r$, the expected cost of agent i will never decrease. If $\mathbf{x}'$ is in Case 1, then there must be an agent j with $x_j = b' \geq \frac{x_r}{3}$. It follows that $c' \leq \max\{|x_l|, x'_r - b'\} \leq \max\{|x_l|, x_r + \Delta - \frac{x_r}{3}\}$, and thus $c' - d \leq \Delta$, $|\frac{x'_r+c'}{2} - \frac{x_r+d}{2}| \leq \Delta$. By the probability distribution, the expected cost of agent i will never decrease.

Therefore, no agent can gain by misreporting, which establishes the proof. □

In the following we prove the approximation ratio, by a case discussion of which value c or d takes.

Theorem 3.7. *For the maximum cost objective, Mechanism 2 is randomized, strategy-proof, and 2.75-approximation.*

Proof. The strategy-proofness is given by Lemma 3.6. For the approximation ratio, we only consider the case $|x_l| \leq x_r$, as the other case $|x_l| > x_r$ is symmetric. By Proposition 3.1, the optimal maximum cost is $OPT(\mathbf{x}) = \max\{|x_l|, b, \frac{x_r - l}{2}\}$. Let $f(\mathbf{x}) = (x_l, y)$ be the output of the mechanism. Clearly, the cost of each agent i with $x_i \leq \frac{x_r}{3}$ is at most $|x_i| \leq \max\{|x_l|, b\} \leq OPT(\mathbf{x})$. It remains to consider the cost of each agent i with $x_i \geq l(\mathbf{x}) > \frac{x_r}{3}$. Let Δ be the maximum cost of such agents, and it suffices to prove that $\mathbb{E}[\Delta] \leq \frac{11}{4} \cdot OPT(\mathbf{x})$. We discuss two cases that correspond to the two cases in the definition of Mechanism 2.

Case 1. $l(\mathbf{x}) \geq \frac{2x_r}{3}$. If $c = |x_l|$, it implies $c \geq \frac{2x_r}{3}$. Since $x_i \geq l(\mathbf{x})$, the distance of agent i to y is at most $\frac{x_r}{3} \leq c = |x_l|$ for every realization of the probability distribution. Therefore, we have $\mathbb{E}[\Delta] \leq |x_l| + |x_l| \leq 2 \cdot OPT(\mathbf{x})$. If $c = x_r - b \leq l(\mathbf{x})$, w.p. $\frac{1}{4}$ we have $y = c$ and $\Delta \leq cost(f(\mathbf{x}), x_r) \leq x_r - y + |x_l| = b + |x_l|$; w.p. $\frac{1}{2}$, $y = x_r$, and $\Delta \leq cost(f(\mathbf{x}), l(\mathbf{x})) \leq y - l(\mathbf{x}) + |x_l| \leq b + |x_l|$; w.p. $\frac{1}{4}$, $y = \frac{c+x_r}{2}$, and $\Delta \leq \frac{x_r - c}{2} + |x_l| = \frac{b}{2} + |x_l|$. Hence, we have $\mathbb{E}[\Delta] \leq \frac{7b}{8} + |x_l| \leq \frac{15}{8} \cdot OPT(\mathbf{x})$, as desired.

If $|x_l| < c = l(\mathbf{x}) < x_r - b$, w.p. $\frac{1}{4}$, $y = c$ and $\Delta \leq cost(f(\mathbf{x}), x_r) \leq x_r - c + |x_l|$; w.p. $\frac{1}{2}$, $y = x_r$ and $\Delta \leq cost(f(\mathbf{x}), l(\mathbf{x})) \leq x_r - c + |x_l|$; w.p. $\frac{1}{4}$, $y = \frac{c+x_r}{2}$ and $\Delta \leq \frac{x_r - c}{2} + |x_l|$. So we have

$$\begin{aligned}\mathbb{E}[\Delta] &\leq \frac{7}{8} \cdot (x_r - c) + |x_l| \\ &\leq \frac{7}{4} \cdot \frac{x_r - l(\mathbf{x})}{2} + |x_l| \leq \frac{11}{4} \cdot OPT(\mathbf{x}).\end{aligned}$$

Case 2. $l(\mathbf{x}) < \frac{2x_r}{3}$. If $d = |x_l| \geq \frac{2x_r}{3}$, for every realization of the probability distribution and for every agent considered (i.e., $x_i \geq l(\mathbf{x})$), it must satisfy $\min\{|x_i|, |x_i - y|\} \leq \frac{x_r}{2} \leq |x_l|$, that is, both the distances to y and to 0 are no more than $|x_l|$. Then we have $\mathbb{E}[\Delta] \leq |x_l| + |x_l| \leq 2 \cdot OPT(\mathbf{x})$.

If $d = \frac{2x_r}{3}$, w.p. $\frac{1}{4}$, we have $y = d$ and $\Delta \leq x_r - d + |x_l|$; w.p. $\frac{1}{2}$, $y = x_r$ and $\Delta \leq x_r - l(\mathbf{x}) + |x_l|$; w.p. $\frac{1}{4}$, $y = \frac{d+x_r}{2}$ and $\Delta \leq \frac{d+x_r}{2} - l(\mathbf{x}) + |x_l|$. Hence, we have

$$\begin{aligned}\mathbb{E}[\Delta] \leq& \frac{1}{4} \cdot (x_r - d) + \frac{1}{2} \cdot (x_r - l(\mathbf{x})) \\ &+ \frac{1}{4} \cdot \left(\frac{x_r + d}{2} - l(\mathbf{x})\right) + |x_l| \\ =& \frac{7}{4} \cdot \frac{x_r - l(\mathbf{x})}{2} + |x_l| \\ \leq& \frac{11}{4} \cdot OPT(\mathbf{x}),\end{aligned}$$

which completes the proof. □

We end this section by providing a lower bound for randomized strategy-proof mechanisms. The instance construction follows from that in proof of Theorem 3.4 in [22] for facility location problems.

Theorem 3.8. *No randomized strategy-proof mechanism has approximation ratio less than $\frac{3}{2}$ for the maximum cost.*

Proof. Let f be a randomized strategy-proof mechanism. Consider a location profile $\mathbf{x} = (L, L+1)$ for some large $L > 0$. We have that $f(\mathbf{x}) = \mathbf{P}$, where $\mathbf{P}$ is a distribution over $\mathbb{R}^2$. It is easy to see that either agent 1 or agent 2 has a cost at least $\frac{1}{2}$. Assume w.l.o.g. that $cost(\mathbf{P}, x_2) \geq \frac{1}{2}$.

Now consider location profile $\mathbf{x}' = (L, L+2)$. Let $f(\mathbf{x}') = (a', b')$ with $a' \leq b'$ be the outcome, where $a', b' \in \mathbb{R}$ are random variables. By the strategy-proofness, it must be that $\mathbb{E}[|x_2 - b'| + |a'|] \geq \frac{1}{2}$, otherwise in instance $\mathbf{x}$ agent 2 can gain by deviating from x_2 to x'_2. Hence, in instance $\mathbf{x}'$, the expected maximum cost is at least $\frac{3}{2}$, while the optimal maximum cost is 1. Thus the ratio is at least $\frac{3}{2}$. □

4 MINIMIZING THE SOCIAL COST

This section is devoted to the social cost objective. While a loose upper bound n for deterministic mechanisms is given by Mechanism 1, we propose a randomized one (Mechanism 3) with a constant performance guarantee.

4.1 Deterministic Mechanisms

We present upper and lower bounds as follows.

Theorem 4.1. *Mechanism 1 is n-approximation for the social cost objective.*

Proof. We assume w.l.o.g. that there exists an optimal solution $(0, y^*)$ with $0 < y^* \leq x_r$. Let (N_1, N_2, N_3) be a partition of the agent set N, where $N_1 = \{i \in N \mid x_i \leq 0\}$, $N_2 = \{i \in N \mid 0 < x_i \leq \frac{y^*}{2}\}$ and $N_3 = \{i \in N \mid x_i > \frac{y^*}{2}\}$. The agents in N_3 can decrease the distance to 0 by using edge $(0, y^*)$, while agents in N_1 and N_2 cannot gain. Then the optimal social cost is

$$OPT(\mathbf{x}) = \sum_{i \in N_1 \cup N_2} |x_i| + \sum_{i \in N_3} |x_i - y^*|.$$

We have

$$\frac{sc(f(\mathbf{x}),\mathbf{x})}{OPT(\mathbf{x})} \le \frac{\sum_{i\in N_1\cup N_2}|x_i| + \sum_{i\in N_3}(|x_r - x_i| + |x_l|)}{\sum_{i\in N_1\cup N_2}|x_i| + \sum_{i\in N_3}|y^* - x_i|} \le \frac{n\cdot|x_l| + \sum_{i\in N_3}|x_r - x_i|}{|x_l| + \sum_{i\in N_3}|y^* - x_i|} \le \frac{n\cdot|x_l| + |N_3|\cdot\max_{i\in N_3}|x_r - x_i|}{|x_l| + \max_{i\in N_3}|x_r - x_i|} \le n,$$

where the third inequality comes from the fact that $\sum_{i\in N_3}|y^*-x_i| \ge |x_r - y^*| + |x_i - y^*| \ge |x_r - x_i|$ for any $i \in N_3$. □

We notice that the approximation ratio n of Mechanism 1 is achievable by an instance with n agents' location profile

$$\mathbf{x} = (-L, 2L, 2L, \ldots, 2L).$$

The optimal solution $(2L, 0)$ has a social cost of L, while the solution $(2L, -L)$ returned by Mechanism 1 has a social cost of $|-L| + (n-1)L = nL$. The ratio is $\frac{nL}{L} = n$.

Next, we prove a lower bound.

Theorem 4.2. *No deterministic strategy-proof mechanism has approximation ratio less than $\frac{3}{2}$ for the social cost.*

Proof. Consider the agents' location profile $\mathbf{x} = (-1, 1)$. Let f be a deterministic strategy-proof mechanism. Assume without loss of generality that $f(\mathbf{x})$ facilitates agent 1. Because a solution can only facilitate agents on a single side, agent 2 must go to the facility directly and have a cost of 1.

Suppose agent 2 misreports her location as $x'_2 = \frac{3}{2}$, and the location profile becomes $\mathbf{x}' = (-1, \frac{3}{2})$. Let $f(\mathbf{x}') = (y_1, y_2)$ with $y_1 \le y_2$. The optimal solution $(0, \frac{3}{2})$ has a social cost of 1. By the approximation ratio, the social cost of $f(\mathbf{x}')$ should be less than $\frac{3}{2}$, and thus $f(\mathbf{x}')$ must facilitate agent 2. Thus, under instance $\mathbf{x}'$, agent 1 has a cost of 1, and agent 2 has cost $|y_2 - \frac{3}{2}| + |y_1| < \frac{1}{2}$. It follows that $|y_2 - 1| + |y_1| < 1$.

Now consider instance $\mathbf{x}$. Under the solution $f(\mathbf{x}')$, agent 2 has a cost $|y_2 - 1| + |y_1| < 1$. Therefore, after misreporting x'_2, agent 2 in $\mathbf{x}$ can decrease her cost, a contradiction. □

4.2 Randomized Mechanisms

The following randomized mechanism adds an edge that connects each agent's location with the facility location 0, with a probability proportional to her distance from 0. It is proven to be strategy-proof and 6-approximation.

Mechanism 3. *Given location profile $\mathbf{x}$, for each agent $k \in N$ with $x_k \neq 0$, return $(0, x_k)$ with probability $\frac{|x_k|}{\sum_{i\in N}|x_i|}$.*

In the following lemma we show the strategy-proofness.

Lemma 4.3. *Mechanism 3 is strategy-proof.*

Proof. For an arbitrary agent $k \in N$, we show that she cannot gain by misreporting. W.l.o.g. assume that k is to the right of 0, i.e., $x_k > 0$. Define $N_0 = \{i \in N \mid x_i \le 0\}, N_1 = \{i \in N \mid 0 < x_i < x_k\}$ and $N_2 = \{i \in N\backslash\{k\} \mid x_i \ge x_k,\}$. Define $D_0 = \sum_{i\in N_0}|x_i|$, $D_1 = \sum_{i\in N_1} x_i$, $D_2 = \sum_{i\in N_2} x_i$, and $D = \sum_{i\in N\backslash\{k\}}|x_i| = D_0 + D_1 + D_2$.

When reporting truthfully, the cost of agent k is

$$cost(f(\mathbf{x}), x_k) = \frac{D_0}{D + x_k}x_k + \sum_{i\in N_1}\frac{x_i}{D + x_k}(x_k - x_i) + \sum_{i\in N_2}\frac{x_i}{D + x_k}\min\{x_k, x_i - x_k\}.$$

Suppose that agent k misreports x'_k so that the location profile becomes $\mathbf{x}' = (\mathbf{x}_{-k}, x'_k)$. We discuss three cases: $0 \le x'_k \le x_k$, $x'_k > x_k$ and $x'_k < 0$.

Case 1. $0 \le x'_k \le x_k$**.** The cost of agent k becomes

$$cost(f(\mathbf{x}'), x_k) = \frac{D_0\cdot x_k}{D + x'_k} + \sum_{i\in N_1}\frac{x_i}{D + x'_k}(x_k - x_i) + \frac{x'_k}{D + x'_k}(x_k - x'_k) + \sum_{i\in N_2}\frac{x_i}{D + x'_k}\min\{x_k, x_i - x_k\}.$$

After a simple computation, we have

$$\begin{aligned} &cost(f(\mathbf{x}'), x_k) - cost(f(\mathbf{x}), x_k)\\ \ge\ &(\frac{D_0}{D + x'_k} - \frac{D_0}{D + x_k})x_k + \frac{x'_k}{D + x'_k} - \frac{x_k}{D + x_k} +\\ &(\frac{D_1}{D + x'_k} - \frac{D_1}{D + x_k}) + (\frac{D_2}{D + x'_k} - \frac{D_2}{D + x_k})\\ =\ &\frac{D_0 x_k + D_1 + D_2 + x'_k}{D + x'_k} - \frac{D_0 x_k + D_1 + D_2 + x_k}{D + x_k}\\ \ge\ &0. \end{aligned}$$

So agent k cannot gain.

Case 2. $x'_k > x_k$**.** The cost of agent k becomes

$$cost(f(\mathbf{x}'), x_k) = \frac{D_0\cdot x_k}{D + x'_k} + \sum_{i\in N_1}\frac{x_i}{D + x'_k}(x_k - x_i) + \frac{x'_k\min\{x_k, x'_k - x_k\}}{D + x'_k} + \sum_{i\in N_2}\frac{x_i\min\{x_k, x_i - x_k\}}{D + x'_k}.$$

We have

$$\begin{aligned} &cost(f(\mathbf{x}'), x_k) - c(f(\mathbf{x}), x_k)\\ \ge\ &x_k(\frac{D_0}{D + x'_k} - \frac{D_0}{D + x_k} + \frac{D_1}{D + x'_k} - \frac{D_1}{D + x_k} + \frac{D_2}{D + x'_k} - \frac{D_2}{D + x_k}) +\\ &\frac{x'_k}{D + x'_k}\min\{x_k, x'_k - x_k\}. \end{aligned}$$

If $x_k \le x'_k - x_k$, then

$$\begin{aligned} &cost(f(\mathbf{x}'), x_k) - cost(f(\mathbf{x}), x_k)\\ \ge\ &x_k(\frac{D}{D + x'_k} - \frac{D}{D + x_k} + \frac{x'_k}{D + x'_k})\\ \ge\ &x_k(\frac{D}{D + x'_k} - \frac{D}{D + x_k} + \frac{x'_k}{D + x'_k} - \frac{x_k}{D + x_k})\\ =\ &0. \end{aligned}$$

If $x_k > x'_k - x_k$, then

$$\begin{aligned}
&cost(f(\mathbf{x}'), x_k) - cost(f(\mathbf{x}), x_k) \\
&\geq x_k\left(\frac{D}{D+x'_k} - \frac{D}{D+x_k}\right) + \frac{x'_k}{D+x'_k}(x'_k - x_k) \\
&= \frac{Dx_k^2 - Dx_kx'_k + (x'^2_k - x_kx'_k)(D+x_k)}{(D+x_k)(D+x'_k)} \\
&\geq \frac{D\cdot(x_k^2 + x'^2_k - 2x_kx'_k)}{(D+x_k)(D+x'_k)} \\
&\geq 0.
\end{aligned}$$

Case 3. $x'_k < 0$**.** When $|x'_k| \leq x_k$, it is easy to see that the cost of agent k is at least that in Case 1. When $|x'_k| > x_k$, the cost of agent k is at least that in Case 2. So agent k has no incentive to lie.

Therefore, combining the above three cases, agent k cannot decrease the cost by misreporting, which establishes the proof. □

Next, we analyze the performance guarantee.

Theorem 4.4. *Mechanism 3 is randomized, strategy-proof, and* 6*-approximation for the social cost objective.*

Proof sketch. Recall from Proposition 2.2 that there must be an optimal solution $(0, y)$ for some $y \in \mathbb{R}$. Assume w.l.o.g. that $y > 0$. For any point $v \in \mathbb{R}$, let $C(v) = sc((0, v), \mathbf{x})$ be the social cost with respect to location profile $\mathbf{x}$ and solution $(0, v)$. Let $D = \sum_{k\in N} |x_k|$ be the total distance of all agents to facility. The approximation ratio of Mechanism 3 is

$$\sum_{k\in N} \frac{|x_k|}{D}\cdot\frac{C(x_k)}{C(y)}.$$

Define $N_1 = \{k \in N : x_k \leq \frac{y}{2}\}$, $N_2 = \{k \in N : x_k \in (\frac{y}{2}, y]\}$, $N_3 = \{k \in N : x_k \in (y, 2y]\}$, and $N_4 = \{k \in N : x_k > 2y\}$. Then the approximation ratio can be written as

$$\sum_{i=1}^{4}\sum_{k\in N_i} \frac{|x_k|}{D}\cdot\frac{C(x_k)}{C(y)}.$$

The proof consists of three steps. In step 1, we prove $\sum_{k\in N_1\cup N_4} \frac{|x_k|}{D}\cdot\frac{C(x_k)}{C(y)} \leq 2$. In step 2, we prove $\sum_{k\in N_2} \frac{|x_k|}{D}\cdot\frac{C(x_k)}{C(y)} \leq 2$. In step 3, we prove $\sum_{k\in N_3} \frac{|x_k|}{D}\cdot\frac{C(x_k)}{C(y)} \leq 2$. Combining these three steps gives the proof. □

We end this section by providing a lower bound for randomized strategy-proof mechanisms. We notice that the strategy-proofness is equivalent to the *partial group strategy-proofness* for the facility location problems in which agents report private locations [16] , which means that for any group of agents at the same location, each individual cannot benefit if they misreport simultaneously.

Theorem 4.5. *No randomized strategy-proof mechanism has approximation ratio less than* 1.02 *for social cost objective.*

Proof. Consider the location profile $\mathbf{x}$ of 9 agents, where 5 agents are located at 0.8, and 4 agents are located at 2. The optimal solution is $(0, 2)$, and thus the optimal social cost is $OPT(\mathbf{x}) = 5 * 0.8 = 4$. Suppose for contradiction that f is a randomized strategy-proof mechanism with an approximation ratio $r < 1.02$. Let $f(\mathbf{x}) = (a, b)$ be the random solution with $a \leq b$, and let p be the probability that $b \geq 1.6$. If $p < 0.9$, then the social cost is at least $p \cdot OPT(\mathbf{x}) + (1-p)\cdot 4.8 = 4.8 - 0.8p \geq 4r$ (where 4.8 is the best possible social cost when $b < 1.6$, attained by solution $(0, 0.8)$), a contradiction to the approximation ratio. So it must be $p \geq 0.9$, and thus the cost of the agents located at 0.8 is at least $p \cdot 0.8 \geq 0.72$.

Now we consider a deviation of the agents located at 0.8 to $x' = 1$, and the location profile becomes $\mathbf{x}'$, where 5 agents are located at 1, and 4 agents are located at 2. Let $f(\mathbf{x}') = (a', b')$ with $a' \leq b'$. The optimal solution $(0, 1)$ has a social cost of 4. By the approximation ratio, the social cost of $f(\mathbf{x}')$ is at most $4r < 4.08$. Let q be the probability that $|b - 0.8| + |a| < 0.3$. If $q < 0.2$, then the social cost is at least $q \cdot OPT(\mathbf{x}') + (1-q)\cdot 4.1 = 4.1 - 0.1q > 4.08$ (where 4.1 is the best possible social cost when $|b - 0.8| + |a| \geq 0.3$, attained by solution $(0, 1.1)$), a contradiction to the approximation ratio. So it must be $q \geq 0.2$, and thus the cost of the agents located at 0.8 is at most $q \cdot 0.3 + (1-q)\cdot 0.8 = 0.8 - 0.5q \leq 0.7 < 0.72$. Hence, the agents located at 0.8 have incentive to misreport their locations as $x'_2 = 1$ simultaneously, which contradicts the partial group strategy-proofness, and thus contradicts the strategy-proofness. □

We remark that the lower bound can be improved slightly if we design the parameters of the instances constructed in the above proof more carefully.

5 CONCLUSION

We have investigated the approximate mechanism design for facility location problems (FLPs) from the structural modification perspective. We focus on a variant of FLPs where the facility is fixed on a real line and the planner aims to add a single costless short-cut edge to reduce the costs/distances of the agents to the facility. There are many open problems and conjectures that are related to our variant. A truly intriguing gap is the one between the deterministic strategy-proof upper bound of *n* and the lower bound of 1.5 for the social cost. This problem seems quite elusive, and we conjecture that it is possible to obtain a lower bound of $\Omega(n)$. Even though other bounds are constant, there are gaps between upper and lower bounds. The most interesting open question is how the analysis for our problem extends to a setting with multiple facilities or multiple new edges to add. It is also interesting to design strategy-proof edge-addition mechanisms for other spaces (e.g., $\mathbb{R}^2$) or discrete networks (e.g., path and tree).

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REFERENCES

[1] Noga Alon, Michal Feldman, Ariel D. Procaccia, and Moshe Tennenholtz. 2010. Strategyproof Approximation of the Minimax on Networks. *Mathematics of Operations Research* 35, 3 (2010), 513–526.

[2] Eleftherios Anastasiadis and Argyrios Deligkas. 2018. Heterogeneous facility location games. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS)*. 623–631.

[3] Oded Berman, Divinagracia I Ingco, and Amedeo R Odoni. 1992. Improving the location of minisum facilities through network modification. *Annals of Operations Research* 40, 1 (1992), 1–16.

[4] Davide Bilò, Luciano Gualà, and Guido Proietti. 2012. Improved approximability and non-approximability results for graph diameter decreasing problems. *Theoretical Computer Science* 417 (2012), 12–22.

[5] Duncan Black. 1948. On the rationale of group decision-making. *Journal of Political Economy* 56, 1 (1948), 23–34.

[6] Hau Chan, Aris Filos-Ratsikas, Bo Li, Minming Li, and Chenhao Wang. 2021. Mechanism Design for Facility Location Problems: A Survey. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI)*. 4356–4365.

[7] Xujin Chen, Xiaodong Hu, Xiaohua Jia, Minming Li, Zhongzheng Tang, and Chenhao Wang. 2018. Mechanism Design for Two-Opposite-Facility Location Games with Penalties on Distance. In *International Symposium on Algorithmic Game Theory (SAGT)*. Springer, 256–260.

[8] Xujin Chen, Xiaodong Hu, Zhongzheng Tang, and Chenhao Wang. 2021. Tight efficiency lower bounds for strategy-proof mechanisms in two-opposite-facility location game. *Inform. Process. Lett.* 168 (2021), 106098.

[9] Erik D Demaine and Morteza Zadimoghaddam. 2010. Minimizing the diameter of a network using shortcut edges. In *Scandinavian Workshop on Algorithm Theory*. Springer, 420–431.

[10] Elad Dokow, Michal Feldman, Reshef Meir, and Ilan Nehama. 2012. Mechanism Design on Discrete Lines and Cycles. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*. 423–440.

[11] Michal Feldman and Yoav Wilf. 2013. Strategyproof facility location and the least squares objective. In *Proceedings of the 14th ACM Conference on Electronic Commerce (EC)*. 873–890.

[12] Chi Kit Ken Fong, Minming Li, Pinyan Lu, Taiki Todo, and Makoto Yokoo. 2018. Facility location games with fractional preferences. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*. 1039–1046.

[13] Dimitris Fotakis and Christos Tzamos. 2013. Strategyproof facility location for concave cost functions. In *Proceedings of the 14th ACM Conference on Electronic Commerce (EC)*. 435–452.

[14] Dimitris Fotakis and Christos Tzamos. 2014. On the power of deterministic mechanisms for facility location games. *ACM Transactions on Economics and Computation (TEAC)* 2, 4 (2014), 1–37.

[15] Chung-Lun Li, S Thomas McCormick, and David Simchi-Levi. 1992. On the minimum-cardinality-bounded-diameter and the bounded-cardinality-minimum-diameter edge addition problems. *Operations Research Letters* 11, 5 (1992), 303–308.

[16] Pinyan Lu, Xiaorui Sun, Yajun Wang, and Zeyuan Allen Zhu. 2010. Asymptotically optimal strategy-proof mechanisms for two-facility games. In *Proceedings of the 11th ACM Conference on Electronic Commerce (EC)*. 315–324.

[17] Pinyan Lu, Yajun Wang, and Yuan Zhou. 2009. Tighter bounds for facility games. In *Proceeding of the 5th International Workshop of Internet and Network Economics (WINE)*. 137–148.

[18] Adam Meyerson and Brian Tagiku. 2009. Minimizing average shortest path distances via shortcut edge addition. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*. Springer, 272–285.

[19] Hervé Moulin. 1980. On strategy-proofness and single peakedness. *Public Choice* 35, 4 (1980), 437–455.

[20] Manos Papagelis, Francesco Bonchi, and Aristides Gionis. 2011. Suggesting ghost edges for a smaller world. In *Proceedings of the 20th ACM International Conference on Information and Knowledge Management (CIKM)*. 2305–2308.

[21] Senni Perumal, Prithwish Basu, and Ziyu Guan. 2013. Minimizing eccentricity in composite networks via constrained edge additions. In *IEEE Military Communications Conference (MILCOM)*. IEEE, 1894–1899.

[22] Ariel D Procaccia and Moshe Tennenholtz. 2013. Approximate mechanism design without money. *ACM Transactions on Economics and Computation (TEAC)* 1, 4 (2013), 1–26.

[23] James Schummer and Rakesh V. Vohra. 2002. Strategy-proof Location on a Network. *Journal of Economic Theory* 104, 2 (2002), 405 – 428.

[24] Paolo Serafino and Carmine Ventre. 2016. Heterogeneous facility location without money. *Theoretical Computer Science* 636 (2016), 27–46.

[25] Jianzhong Zhang, Zhenhong Liu, and Zhongfan Ma. 2000. Some reverse location problems. *European Journal of Operational Research* 124, 1 (2000), 77–88.