Debt Transfers in Financial Networks: Complexity and Equilibria

Panagiotis Kanellopoulos
University of Essex
Colchester, United Kingdom
panagiotis.kanellopoulos@essex.ac.uk

Maria Kyropoulou
University of Essex
Colchester, United Kingdom
maria.kyropoulou@essex.ac.uk

Hao Zhou*
University of Essex
Colchester, United Kingdom
hz18065@essex.ac.uk

ABSTRACT

We consider the operation of debt transfer in interbank networks. In particular, assuming a financial system that is represented by a network of banks and their bilateral debt contracts, we consider the setting where a bank can transfer the right to claim a debt to one of its lenders, under some assumptions. Perhaps surprisingly, such an operation can benefit the banks involved, and potentially the entire network as well, in terms of maximizing natural objectives related to financial well-being, like total assets and equity.

We consider debt transfers in both a centralized and a distributed (game-theoretic) setting. First, we examine the computational complexity of computing debt transfer combinations that maximize total payments or total equity, or satisfy other desirable properties. We then study debt transfer operations from a game-theoretic standpoint. We formally define games that emerge when banks can be strategic about choosing whether or not to transfer their debt claims. We prove theoretical results on the existence and quality of pure Nash equilibria in debt transfer games, as well as the computational complexity of relevant problems. We complement our theoretical study with an empirical analysis involving different heuristics about computing debt transfer combinations, as well as game-playing dynamics of debt transfer operations on synthetic data.

KEYWORDS

Financial networks; Debt transfers; Nash equilibria; Price of anarchy

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1 INTRODUCTION

Loan assignments are means by which a lender can transfer its interest in a loan to another lender. For example, if someone has a right, e.g., to claim damages, against someone else, they can transfer that right to a third party. In this paper, we consider loan assignments that can be used to cancel out other debts and study their potential to improve the well-being of financial systems.

A financial system is represented by a network, where nodes correspond to banks, or other institutions that engage in financial transactions. Bilateral financial obligations between banks can be captured by directed labeled edges, where the label corresponds to the amount of the actual debt/liability. The total assets of each bank consist of a fixed amount of external assets (not affected by the network) and potential incoming payments. Banks utilize their total assets to cover their liabilities by making payments to their lenders. If a bank's assets are not enough to cover its liabilities, that bank will be in default and the value of its assets will be decreased (e.g., by liquidation); the extent of this decrease is captured by default costs and essentially implies that the corresponding bank will have only a part of its total assets available for making payments. For any financial network, it is possible to compute clearing payments, i.e., payments that satisfy the following three principles of bankruptcy law (see, e.g., [7]): i) absolute priority, i.e., banks must first pay their liabilities in full in order to have positive equity, ii) limited liability, i.e., banks cannot pay more than their total assets, and iii) proportionality, i.e., in case of default, payments to lenders are made in proportion to the respective liability.

We use the term *debt transfer* to refer to an operation where a bank can choose to transfer the right it has to claim a debt to one of its lenders, if that would alleviate its own debt to that lender. In particular, if a bank B is owed a certain amount and at the same time B owes the same amount to some other bank, then B can decide to replace these two loans with a single loan from its borrower to its lender. Loan assignments are governed by law and have been considered extensively in the legal literature (see e.g., [1, 4]). To the best of our knowledge, loan assignments have not been considered in the scientific literature. Motivating examples can be encountered commonly in the retail/supply chain market where purchasing with credits is very common: if A buys something from B with credit and then B buys something from C with credit, then these purchasing activities can result in a single payment obligation from A to C (assuming that both purchases have the same monetary value). Another example is using vouchers to repay a debt: Consider for example an Amazon voucher as a right to claim products of a certain amount from Amazon. In other words, an Amazon voucher can be thought of as a debt obligation from Amazon to some party B that has purchased the voucher. Now if B has another debt towards some third party C, they can repay that debt by offering the voucher, i.e., B transfers to their lender C, their right to claim from Amazon.

There are several cases where debt transfers can be beneficial to the corresponding bank. This might seem counterintuitive, as a debt transfer directly reduces a bank's income and, at best, the bank will just write off an equivalent debt. It is, however, true that transferring one's debt can improve the well-being of the entire network, which may lead to fewer banks in default incurring associated costs. This may lead to an increased cash flow through the network that will benefit many banks, including the one that made the original debt transfer. In this paper we aim to get a better understanding of the potential of debt transfer operations towards improving the well-being of a financial system.

*Corresponding author

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Debt transfer operations can be meaningful from a regulator's perspective who can potentially enforce them in order to achieve some objectives. In this context the goal would be to compute a collection of debt transfers that achieve certain objectives related to the financial well-being of the system. Natural metrics of the system's financial well-being include sum of total assets (equivalent to sum of total payments also known as total liquidity) and total equity, where the equity of a bank reveals the assets that it has available after making payments, if any (and is equal to zero for banks in default). Debt transfer operations can also be meaningful from a game theoretic perspective as banks can be strategic about transferring their debt claims. Our work considers both the centralized and distributed approach and performs a theoretical and empirical analysis of related questions.

Roadmap and summary of results. We study debt transfer operations in financial networks, where banks can transfer their right to claim debts to their lenders. We begin with some preliminary definitions in Section 2. In Section 3 we consider the computational complexity of selecting a collection of debt transfers that optimizes certain objectives, e.g., maximizing total payments or equity. In Section 4 we introduce debt transfer games that emerge when banks can strategically transfer their debt claims. Our game-theoretic analysis considers two different definitions of utility motivated by the financial literature, namely total assets, or equity, respectively. We analyze each variant with respect to the existence, computational complexity and quality of the Nash equilibria that arise. Specifically, we show a network without pure Nash equilibria when banks wish to maximize their total assets and default costs are applied, while there always exists a Nash equilibrium in games where players wish to maximize their equity and no default costs apply. In terms of quality, we prove that Nash equilibria can have arbitrarily worse social welfare than the optimal state, while they can have arbitrarily better social welfare than the initial network. We also investigate the computational aspects of equilibrium related problems at debt transfer games. In Section 5 we complement our theoretical results with an empirical analysis on synthetic networks. In particular, we examine the performance of simple heuristics for finding a collection of debt transfers according to various performance measures, and we also study the dynamics of game-playing and the quality of associated equilibria.

Overall, our analysis provides evidence supporting the use of debt transfers for improving the financial well-being of a system.

Related work. Eisenberg and Noe [7] introduced a seminal model for financial systems with debt contracts and proportional payments. This base model was later extended by Rogers and Veraart [23] to include default costs, and further additional features have since been introduced, such as cross-ownership relations ([8, 27]) and credit default swaps ([16, 26]).

A large body of recent work considers game-theoretic aspects in financial networks. Papp and Wattenhofer [20] consider the incentives of banks to remove incoming edges, redefine the seniorities of liabilities, as well as to donate external assets, while in [21] they consider the impact of debt swapping in mitigating risk. Kanellopoulos et al. [14] study a game where banks can remove incoming edges and also allow for bailout from a central authority. Bertschinger et al. [3] and Kanellopoulos et al. [13] study strategic behavior under payment schemes other than the proportional one. In very recent work, Hoefer and Wilhelmi [10] consider clearing games with different seniorities. Bertschinger et al. [2] study the existence and structure of equilibria in a game modeling fire sales, as well as the convergence of best-response dynamics. Additionally, Schuldenzucker et al. [24] study the impact of portfolio compression in financial network and derive sufficient conditions leading to a Pareto improvement for all banks.

Schuldenzucker et al. [25] consider the complexity of finding clearing payments when credit default swaps are allowed. In a similar spirit, Ioannidis et al. [11] examine the complexity of the clearing problem in financial networks with derivatives and priorities among creditors, while in [12] they study the clearing problem from the point of view of irrationality and strength of approximation. Papp and Wattenhofer [22] study which banks are in default, and how much of their liabilities these can pay.

Previous studies on simulating financial networks have assumed that the amount of liabilities and the edge degrees in such networks follow a power-law distribution, while the recovery rate and the bank assets follow a bimodal distribution (as in [15]) or the normal distribution (as in [28]). Leventides et al. [17] simulated contagion dynamics in fully random networks based on the uniform distribution, while in [18] a similar approach is followed in terms of liabilities and external assets. Finally, Chen et al. [5] develop a dynamic model to study systemic risk of the banking network, so as to study the dynamics of bank defaults, and provide an analysis on a network of 22 German banks.

2 PRELIMINARIES

Financial networks. A *financial network* $N = (V, E)$ comprises a set $V = \{v_1, \dots, v_n\}$ of n banks, where each bank v_i initially has some non-negative *external assets* e_i; these correspond to income received from entities that are outside the financial system. Banks have *liabilities*, that is, payment obligations due to debt contracts, among themselves. In particular, bank v_i (the borrower) has a liability of l_{ij} to bank v_j (the lender), with $l_{ij} \geq 0$ and $l_{ii} = 0$; note that $l_{ij} > 0$ and $l_{ji} > 0$ may both hold simultaneously. We denote by $L_i = \sum_j l_{ij}$ the total liabilities of v_i. Banks that are able to pay their obligations in full are *solvent*, while those that cannot are *in default* or, also, *insolvent*.

We use p_{ij} to denote the actual payment from v_i to v_j and note that p_{ij} need not equal the liability l_{ij}; we assume that $p_{ii} = 0$. Let $\mathbf{P} = (p_{ij})$ with $i, j \in [n]$[1] be the induced payment matrix and $p_i = \sum_{j \in [n]} p_{ij}$ be the total outgoing payments of v_i. A bank in default may need to liquidate its external assets or make payments to entities outside the financial system (e.g., to pay wages). This is modeled using *default costs* $\alpha \in [0, 1]$. That is, a bank in default may only use an α fraction of its external assets and incoming payments. By the absolute priority and limited liability regulatory principles, discussed in the introduction, a solvent bank must fully pay all its obligations to all its lenders, while a bank in default must repay as much of its debt as possible, taking default costs also into account, and each partial payment is proportional to its total liabilities. Summarizing, it must hold that $\mathbf{P} = \Phi(\mathbf{P})$, where

[1] $[n]$ stands for the set of integers $\{1, \dots, n\}$.

$$\Phi(\mathbf{x})_{ij} = \begin{cases} l_{ij}, & \text{if } L_i \le e_i + \sum_{j=1}^n x_{ji} \\ \alpha \cdot \left(e_i + \sum_{j=1}^n x_{ji}\right) \cdot \frac{l_{ij}}{L_i}, & \text{otherwise.} \end{cases}$$

Payments $\mathbf{P}$ that satisfy these constraints are *clearing payments*; these need not be unique. Maximal clearing payments, i.e., ones that point-wise maximize all corresponding payments, are known to exist [7, 23] and can be computed in polynomial time.

Given $\mathbf{P}$, the *total liquidity* (or *systemic liquidity*) equals the sum of payments, while for each bank v_i, its *total assets* $a_i(\mathbf{P})$ are its external assets and incoming payments, i.e., $a_i(\mathbf{P}) = e_i + \sum_{j\in[n]} p_{ji}$, and its *equity* $E_i(\mathbf{P})$ is defined as $E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}$. When $\mathbf{P}$ is clear from the context, we will simply use a_i and E_i instead.

Debt transfer. A *debt transfer* $<v_j, v_i, v_k>$ is an operation, involving three banks, v_i (the broker), v_j (the borrower), and v_k (the lender) with $l_{ji} = l_{ik}$, where liabilities from v_j to v_i and from v_i to v_k are replaced by a single liability (of equal claim) from v_j to v_k.

An example is presented in Figure 1. Let $\alpha = 1$, observe that $l_{12} = l_{23}$ and consider the financial network arising when these liabilities are replaced by a new one between v_1 and v_3. The clearing payments in the initial network (before the debt transfer) are $p_{12} = 1$, $p_{23} = 2$, $p_{24} = 1$, $p_{42} = 1$ with total assets $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $a_4 = 1$ and equities $E_1 = E_2 = E_4 = 0$, and $E_3 = 2$. After the debt transfer, we have $p'_{13} = 1$, $p'_{24} = 4$, $p'_{42} = 7/2$ with $a'_1 = 1$, $a'_2 = 9/2$, $a'_3 = 1$, $a'_4 = 4$ and $E'_1 = 0$, $E'_2 = 1/2$, $E'_3 = 1$ and $E'_4 = 1/2$. We conclude that v_2 is better off after the debt transfer in terms of both total assets and equity.

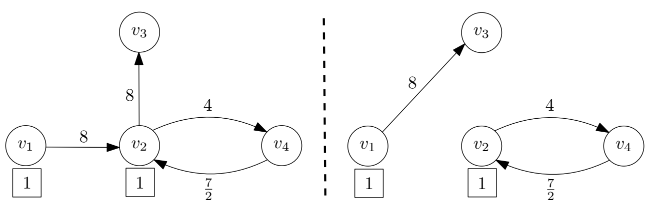

Figure 1: The left subfigure shows the initial network, while the right subfigure shows the network after the debt transfer by v_2. Nodes correspond to banks, edges are labeled with the respective liabilities, while external assets appear in a rectangle near the relevant bank.

Note that a debt transfer $<v_j, v_i, v_k>$ either creates a new liability between v_j and v_k or increases the existing liability. The latter might lead to new possible debt transfers involving v_j, v_k and another bank, where v_k would now be the broker.

Debt transfer games. Given a financial network N, a bank can select to transfer some debt claims to maximize its utility (either total assets or equity). That is, bank v_i can transfer a debt claim from bank v_j to another bank v_k provided that $l_{ji} = l_{ik}$. Given v_i's possible debt transfers, its *strategy* s_i consists in selecting which debt claims to transfer and which to preserve.

Recall that clearing payments are not necessarily unique and it is standard practice (see, e.g., [7, 23]) to focus on maximal clearing payments to avoid this ambiguity. A strategy profile $\mathbf{s}$ is a *Nash equilibrium* if no bank can increase its utility by deviating; again, assuming that the maximal clearing payments will be realized every time. We limit our study to pure Nash equilibria.

Given clearing payments $\mathbf{P}$, the social welfare $SW(\mathbf{P})$ is the sum of the banks' utilities; the particular utility notion (total assets or equity) will be clear from the context. The optimal social welfare is denoted by OPT.

Let $\mathbf{P_{eq}}$ be the set of clearing payments consistent with Nash equilibrium strategy profiles. The price of anarchy (PoA) of a particular instance is defined as the worst-case ratio of the optimal social welfare over the social welfare achieved at any equilibrium at the instance, PoA $= \max_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})}$. In contrast, the price of stability (PoS) of a given instance of a game measures how far the highest social welfare that can be achieved at equilibrium is from the optimal social welfare, i.e., PoS $= \min_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})}$. We also study the effect of anarchy (EoA) and the effect of stability (EoS) that measure the discrepancy between the social welfare SW_N of the original network (no debt transfers) and that of the worst (best, respectively) Nash equilibrium; see also [14]. These are defined as EoA $= \max_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{SW_N}{SW(\mathbf{P})}$ and EoS $= \min_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{SW_N}{SW(\mathbf{P})}$. The *Price of Anarchy* of a game is the maximum PoA of any instance of the given game; similar definitions apply to the Price of Stability, Effect of Anarchy, and Effect of Stability of a game.

3 COMPUTING OPTIMAL DEBT TRANSFERS

In this section, we study how a financial authority (such as a regulator) can exploit debt transfers to affect financial networks. In particular, we are interested in how a suitable collection of debt transfers can lead to *systemic solvency* (i.e., all banks are solvent), or to an increased total liquidity.

We begin with the objective of achieving systemic solvency. Although a series of debt transfers could reduce significantly the number of banks in default (see Figure 2), our first result states that the financial authority cannot use debt transfers to transform a financial network, with at least one bank in default, so that it becomes systemic solvent.

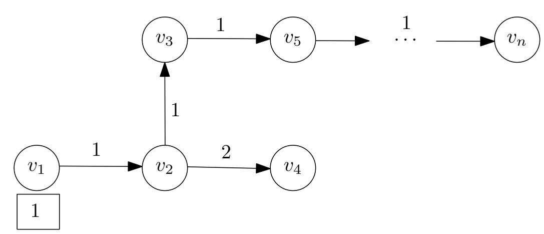

Figure 2: The number of banks in default is reduced from $n-2$ to 1 after v_2 transfers its debt claim.

Theorem 3.1. *A financial network with at least one bank in default cannot be made systemic solvent by debt transfers.*

We now focus on increasing total liquidity, i.e., the total payments that travel through the financial network, and prove that computing an optimal collection of debt transfers is NP-hard, even when there are effectively no default costs. Note that this implies hardness of maximizing the sum of total assets as well, as the latter equals total liquidity plus the (fixed) sum of external assets.

Theorem 3.2. *In networks without default costs, i.e., $\alpha = 1$, it is NP-hard to compute a collection of debt transfers that maximizes total liquidity.*

Proof. The proof relies on a reduction from the NP-complete problem Restricted Exact Cover by 3-Sets (RXC3) [9], a variant of Exact Cover by 3-Sets (X3C). In RXC3, we are given an element set X, with $|X| = 3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.

Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. We add bank t_i for each element i of X, banks v_i, v_i' and v_i'' for each subset i in C, as well as another bank T. Each bank v_i, corresponding to set $(x, y, z) \in C$, has external assets $e_i = 3$ and liability 1 to each of the three banks t_x, t_y, and t_z corresponding to the three elements $x, y, z \in X$, as well as liability M to v_i'', where M is an arbitrarily large number. Furthermore, each v_i' has an external asset of 1 and liability of M to v_i, while all t_i's have liability 1 to T; see also Figure 3. Note that this construction requires polynomial time.

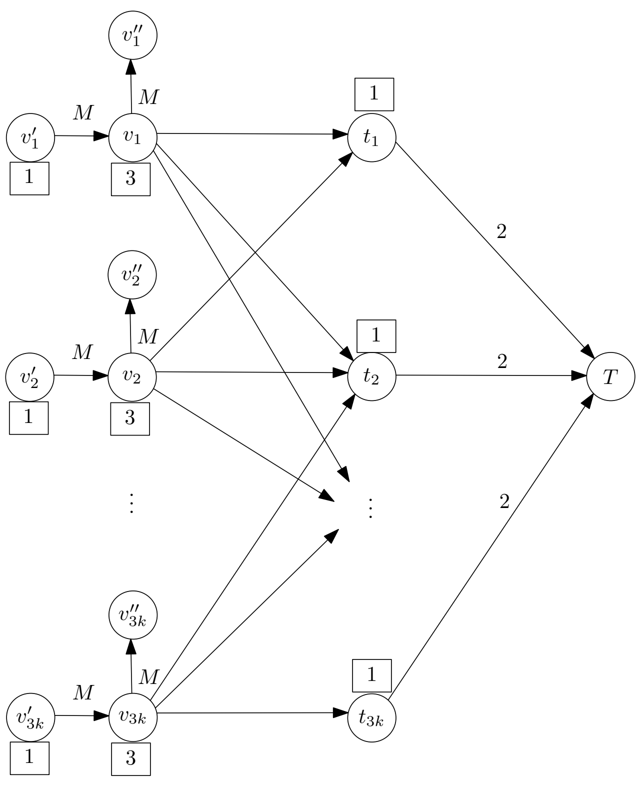

Figure 3: The reduction used in the proof of Theorem 3.2. All edges with missing labels correspond to liability 1.

We first argue that, when systemic liquidity is maximized, no t_i, for $i \in \{1, 2, \ldots, k\}$, makes any debt transfers, as keeping its own debt claim unchanged is weakly better in terms of systemic liquidity. Next, we show that the maximal systemic liquidity of $20k$ can be achieved if and only if instance $\mathcal{I}$ is a 'yes'-instance for problem RXC3.

Let instance $\mathcal{I}$ be a 'yes'-instance for RXC3 and let C' be the solution to $\mathcal{I}$. We claim that $\mathcal{I}'$ admits a solution with systemic liquidity $20k$. Indeed, it suffices to let all v_i's with $i \in C'$ make the debt transfer from v_i' to v_i'', while all other v_i's keep their debt claims unchanged. This choice makes each edge (t_i, T) for $i = \{1, 2, \ldots, 3k\}$ saturated with the following payments. We have $\sum_{i \in C'} p_{v_i', v_i''} = k$ due to the debt transfers, while each v_i, with $i = (x, y, z) \in C'$, has total outgoing payments of 3 to t_x, t_y and t_z, which, taking into account also the external assets in t_x, t_y and t_z, lead to a total payment of 6 to T. Overall, the liquidity emanating from these k v_i's with $i \in C'$ is $9k$. Finally, the payments to and from banks v_i that do not transfer their debt claims are $10k$, as each of the $2k$ such banks receives a payment of 1 and pays 4 to its direct neighbors; hence, the systemic liquidity is $20k$.

It suffices to show that any collection of debt transfers that generates liquidity of at least $20k$ can lead to a solution for instance $\mathcal{I}$. Let χ be the number of agents v_i whose debt claim from v_i' is transferred. We first show that if the liquidity is at least $20k$, then it must be $\chi = k$.

Note that the total liquidity starting from the v_i''s to their neighbors is $3k$, while the total liquidity from all v_i's to their direct neighbors is $3\chi + 4(3k - \chi) = 12k - \chi$. When $\chi < k$, note that the total payments from v_i's to t_i's equal $3\chi + 4 \cdot \frac{3}{M+3}(3k - \chi) < 3\chi + 1$ as M is arbitrarily large. Therefore, the total liquidity from t_i's to T is at most $3\chi + 1 + 3k$ and the systemic liquidity is at most $3k + 12k - \chi + 3\chi + 1 + 3k = 18k + 2\chi + 1 < 20k$ as $\chi < k$. Similarly, when $\chi > k$, the systemic liquidity is at most $3k + 12k - \chi + 6k = 21k - \chi < 20k$ where $3k$ and $12k - \chi$ are the exact liquidity from v_i''s and v_i's to their own outgoing neighbors respectively, while the liquidity from t_i's to T is at most $6k$.

It remains to argue about the case $\chi = k$. If these k banks can cover all t_i's, then we obtain a solution to RXC3; a contradiction. So, there exists at least one bank t_i that receives payment from at least two v_i's and the total liquidity from the t_i's to T would be at most $6k - 1$. The total liquidity in that case would be at most $3k + 11k + 6k - 1 < 20k$. The proof is complete. □

When focusing on the objective of maximizing the total equity, we make use of the following lemma.

Lemma 3.3 ([13, 26]). *In any financial network without default costs, the total equity, after clearing, equals the sum of external assets, that is, $\sum_i E_i = \sum_i e_i$.*

As, according to Lemma 3.3, the sum of equities is always equal to the sum of external assets when there are no default costs, it holds that any collection of debt transfers maximizes the total equity.

Corollary 3.4. *Given a financial network without default costs, any collection of debt transfers maximizes total equity.*

The situation changes drastically, however, when non-trivial default costs apply.

Theorem 3.5. *In financial networks with default costs $\alpha \in (0, 1)$, the following problems are NP-hard:*

a) *computing a collection of debt transfers maximizing the total equity;*

b) *computing a collection of debt transfers minimizing the number of banks in default;*
c) *computing a collection of debt transfers that guarantees that a given bank is no longer in default and minimizes the amount of debt claims transferred.*

4 DEBT TRANSFER GAMES

In this section we consider the distributed, game-theoretic variant, where each bank may decide to transfer some of its debt claims, if applicable, in order to increase its utility. We first consider the case where banks care about their total assets and then we consider the equity; recall that, as the example in Figure 1 shows, a debt transfer can indeed lead to increased utility.

4.1 Maximizing Total Assets

We consider the utility function of total assets and observe that, when non-trivial default costs apply, there exist debt transfer games that do not admit Nash equilibria.

Theorem 4.1. *There exists a debt transfer game with default costs $\alpha \in (0, 1)$ that does not admit Nash equilibria, when banks wish to maximize their total assets.*

We now investigate the quality of equilibria. Although, the social welfare at an equilibrium could be arbitrarily lower than the optimal one, we find that the quality could be much better than that in the initial network in terms of social welfare.

Theorem 4.2. *The Price of Stability in debt transfer games with default costs $\alpha \in [0, 1]$ where banks wish to maximize their total assets is unbounded.*

Theorem 4.3. *The Effect of Anarchy in debt transfer games with default costs $\alpha \in [0, 1]$ where banks wish to maximize their total assets is arbitrarily close to* 0.

Proof. Consider the network N represented in Figure 4 and observe that v_2 is the only bank that can transfer a debt claim. In the

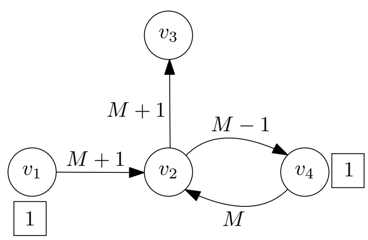

Figure 4: A debt transfer game with default costs $\alpha \in [0, 1]$ that yields Effect of Anarchy arbitrarily close to 0 when players wish to maximize their total assets.

clearing state of the original network, when v_2 has not transferred its debt claim, it is in default as the payment it receives from v_1 is α. In effect, v_4 will also be in default so the payments will satisfy $p_{12} = \alpha$, $p_{42} = \alpha(p_{24} + 1)$, $p_{23} = \alpha(p_{12} + p_{42})\frac{M+1}{2M}$ and $p_{24} = \alpha(p_{12} + p_{42})\frac{M-1}{2M}$. In particular, the payments are $p_{12} = \alpha$, $p_{42} = \frac{(M-1)\alpha^3+2M\alpha}{2M-(M-1)\alpha^2}$, $p_{23} = \frac{2(M+1)\alpha^2}{2M-(M-1)\alpha^2}$ and $p_{24} = \frac{2\alpha^2(M-1)}{2M-(M-1)\alpha^2}$. The sum of total assets equals

$$
\begin{aligned}
SW_N &= \frac{2M\alpha + (M-1)\alpha^3 + 4M\alpha^2}{2M - (M-1)\alpha^2} + 2 + \alpha \\
&\leq \frac{7M-1}{M+1} + 3 \\
&< 10,
\end{aligned}
$$

where the first inequality holds since the expression is an increasing function of $\alpha \in [0, 1]$.

If v_2 transfers its debt claim, then we get clearing payments $p_{13} = \alpha$, $p_{24} = M - 1$ and $p_{42} = M$, yielding a sum of total assets equal to $2M + 1 + \alpha$. As this is the unique Nash equilibrium, we obtain that the Effect of Anarchy is at most $\frac{10}{2M+1+\alpha}$, i.e., it can become arbitrarily close to 0 for sufficiently large M. □

We now consider further questions regarding the complexity of computing equilibria and of deciding on their existence.

Theorem 4.4. *In debt transfer games with default costs $\alpha \in (0, 1)$, where banks wish to maximize their total assets, the following problems are NP-hard:*

a) *computing a Nash equilibrium when one is guaranteed to exist;*
b) *computing the best response;*
c) *deciding if there exists a pure Nash equilibrium.*

4.2 Maximizing Equity

We now shift our focus on the utility function being the bank's equity; we begin by proving existence of Nash equilibria for the setting without default costs.

Theorem 4.5. *In debt transfer games without default costs, where banks wish to maximize their equity, the strategy profile where all banks transfer their debt claims is a Nash equilibrium.*

Proof. Consider a debt transfer game on a financial network S with banks v_q for $q = 1, \ldots, n$. Let S^* denote a state where all eligible debt claims are transferred.[2] Assume for a contradiction that some bank v_i can increase its equity by deviating from S^* to a strategy where v_i does not transfer the set of debts C, which contains at least one of its debt claims that is transferred under S^*; denote the resulting state by S^*_{-i}.

Our proof uses two auxiliary networks, A^* and A^*_{-i}, that are constructed as follows; Figures 5 and 6 show how we can adapt particular parts of S^* and S^*_{-i}, respectively to get A^* and A^*_{-i}.

A^* is constructed by S^* as follows. For each debt $<v_j, v_i, v_k> \in C$ (i.e., that v_i decides not to transfer), we remove edge (v_j, v_k) and add banks u and v with one edge from v_j to v having liability $l_{v_j v} = l_{ji}$ as well as one edge from u to v_k having liability $l_{uv_k} = p_{jk}$, where p_{jk} is the payment from v_j to v_k in the clearing state of S^*. The new banks have external assets $e_u = l_{uv_k} = p_{jk}$ and $e_v = 0$. Observe that by construction, under the clearing state of A^*, each bank v_q, $q = 1, \ldots, n$, will have the exact same equity as they did under S^*.

Our second auxiliary network A^*_{-i} is constructed by S^*_{-i} if for each debt claim $<v_j, v_i, v_k> \in C$ we do the following. We remove

[2]Note that the set of identified eligible debt transfers at the original network S might not be identical to the set of debt transfers that are implemented in S^*, as it is possible that transferring a debt might make another one ineligible or might create a new eligible one. This does not contradict the fact that S^* is the network that will be reached starting from S, if the strategy of each bank is to transfer all eligible debts.

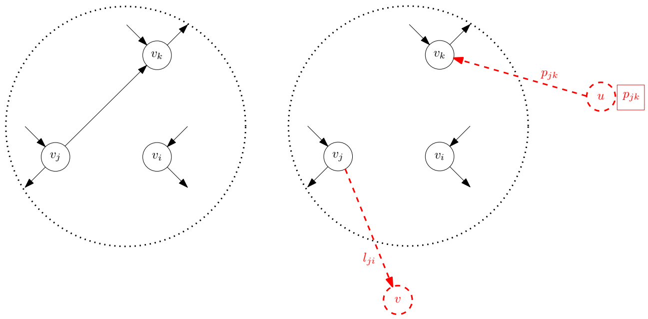

Figure 5: The figure shows how to construct auxiliary network A^* from S^* by focusing on the relevant part.

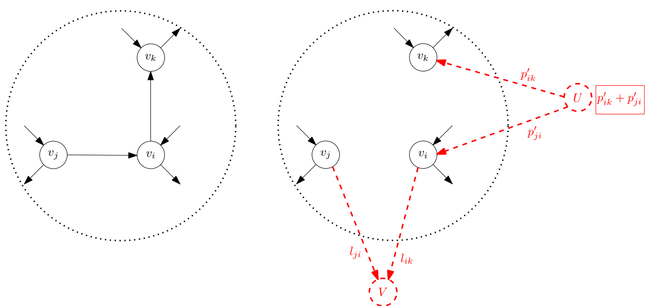

Figure 6: The figure shows how to construct auxiliary network A^*_{-i} from S^*_{-i} by focusing on the relevant part.

edges (v_j, v_i) and (v_i, v_k) and add banks U and V with one edge from v_j to V having liability $l_{v_jV} = l_{ji}$, one edge from v_i to V having liability $l_{v_iV} = l_{ik}$, one edge from U to v_i having liability $l_{Uv_i} = p'_{ji}$, as well as one edge from U to v_k having liability $l_{Uv_k} = p'_{ik}$, where p'_{ji} and p'_{ik} are the corresponding payments in the clearing state of S^*_{-i}. The new banks have external assets $e_U = l_{Uv_i} + l_{Uv_k} = p'_{ji} + p'_{ik}$ and $e_V = 0$. Since, by assumption, bank v_i's deviation from S^* to S^*_{-i} is profitable, it holds that $E_i(S^*_{-i}) > E_i(S^*) \geq 0$, hence v_i fully repays its obligations at the clearing state of S^*_{-i} and $p'_{ik} = l_{ik}$ for each v_k appearing in C. Similarly to before, we observe that by construction, under the clearing state of A^*_{-i}, each bank v_q, $q = 1, \ldots, n$, will have the exact same equity as they did under S^*_{-i}.

By Lemma 3.3, and since no default costs apply, we know that the total equity under the clearing state of both networks S^* and S^*_{-i} equals the sum of the corresponding external assets, hence it is the same. By construction of the two auxiliary networks we conclude that the sum of equities of all banks v_q, $q = 1, \ldots, n$, in A^* and in A^*_{-i} is also the same, i.e.,

$$\sum_q E_q(A^*_{-i}) = \sum_q E_q(A^*) = \sum_q e_q. \tag{1}$$

By assumption of the profitable deviation of v_i from S^* to S^*_{-i}, i.e., $E_i(S^*_{-i}) > E_i(S^*)$, and the equivalence between the equities of respective banks between S^* and A^* as well as S^*_{-i} and A^*_{-i} we have that $E_i(A^*_{-i}) > E_i(A^*) \geq 0$. This implies that v_i has positive equity and, thus, is solvent and can repay all its liabilities ($p'_{ik} = l_{ik} \geq p_{jk}$). So, the incoming payments in A^*_{-i} of each v_k that appears in C are at least equal to the ones in A^*. By considering the propagation of the assets of v_i and the assets of each v_j that appears in C, to the otherwise equivalent networks A^*_{-i} and A^*, we can conclude that $E_q(A^*_{-i}) \geq E_q(A^*)$ for each $q = 1, \ldots, n$ (recall that the inequality is strict for $q = i$). This implies that $\sum_q E_q(A^*_{-i}) > \sum_q E_q(A^*)$; a contradiction to Equality (1). We conclude that no bank can increase its equity by deviating from S^*, so the strategy profile where all banks transfer their debt claims is a pure Nash equilibrium as desired. □

Recall that, by Lemma 3.4, when $\alpha = 1$ the sum of equities is independent of the bank strategies and, hence, the Price of Anarchy and Stability, as well as the Effect of Anarchy and Stability, is 1. When $\alpha < 1$, however, we obtain results on the quality of equilibria that are similar to those when maximizing total assets.

Theorem 4.6. *The Price of Stability in debt transfer games with default costs $\alpha \in [0, 1)$ where banks wish to maximize their equity is unbounded.*

Theorem 4.7. *The Effect of Anarchy in debt transfer games with default costs $\alpha \in [0, 1)$ where banks wish to maximize their equity is arbitrarily close to 0.*

5 EMPIRICAL ANALYSIS

We now present our empirical analysis of debt transfers on synthetic networks. Due to lack of space some parts have been omitted; the complete analysis will appear in the full version.

5.1 Empirical Analysis of the Centralized Case

We examine the performance of different algorithms for computing debt transfer combinations on synthetic networks. Recall that, in Section 3 we saw that it is NP-hard to compute collections of debt transfers that maximize the sum of total assets (or, equivalently, the total liquidity, Theorem 3.2) even in the case without default costs, while for the case with default costs it is NP-hard to compute collections of debt transfers that maximize the total equity or that minimize the number of banks in default (Theorem 3.5). We here check how a rather straightforward approach performs on a set of randomly generated networks, in terms of all the aforementioned objectives and for different values of default costs.

5.1.1 *Experimental Setup for the Centralized Case.*

Network generation. As is common in the literature (see, e.g., [17]), and for simplicity reasons, we have chosen to work with the uniform distribution in various ranges. We construct 1000 networks of $N = 25$ nodes each, corresponding to banks. We consider each of these networks for each of the following default costs values: $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. For each bank, we select its number of outgoing edges (that correspond to liabilities) from a uniform distribution in $[0, 24]$ and create outgoing edges to that many randomly picked other banks. To capture heterogeneity among debts and, in particular, the existence of small, medium and large loans in a financial network, we follow the approach in [17] and randomly pick among ranges $[0, 10]$, $[0, 20]$, and $[0, 35]$ for each liability; we then select a number from the corresponding range as the amount of the corresponding liability. Regarding the external assets each bank holds, we pick a number s uniformly at random from the range $[0, 25]$ and set the externals of s (randomly picked) banks to zero. The amount of external assets of each remaining bank in the network is drawn from a uniform distribution in the range $U(0, 100)$. This approach simulates an external shock to s banks after the creation of the network, as is common in the literature (see, e.g., [17]); it helps us keep realistic aspects like random external assets at financially weak networks.

Our algorithms. We consider the following algorithms/heuristics. We note that the distinction between the two classes of algorithms below is that the latter only allows initially insolvent banks (banks in default) to perform debt transfers.

Random Banks (RB$_x$): Identify all banks with eligible debt transfers. Randomly pick a fraction of $x\%$ of them and execute their debt transfers in an arbitrary order. If a debt transfer is no longer eligible after the execution of previous ones, we skip it. This is defined for $x \in \{25, 50, 75, 100\}$.

Random Insolvent Banks (RIB$_x$): Identify all insolvent banks with eligible debt transfers. Randomly pick a fraction of $x\%$ of them and execute their debt transfers in an arbitrary order. If a debt transfer is no longer eligible after the execution of previous ones, we skip it. This is defined for $x \in \{25, 50, 75, 100\}$.

All Insolvent Banks (AIB): Run RIB$_{100}$ repeatedly until no eligible debt transfer among insolvent banks exists. Keep the order in which the banks are considered consistent across rounds.

Evaluation metrics. We evaluate the performance of the above algorithms according to four different criteria, namely total liquidity, total equity, number of insolvent banks, and total recovery rate. The ratios $\mathcal{R}_{TL}$, $\mathcal{R}_{TE}$, $\mathcal{R}_{IB}$, and $\mathcal{R}_{RR}$ compare the corresponding values at the original network N, and the network N' that emerges after the execution of the algorithm. Formal definitions appear at the supplementary material.

5.1.2 Results. In our results, we display the trimmed mean of corresponding datasets, that is, we calculate the mean of the data after discarding outliers. This widely-used approach (e.g., see [6, 19]) allows us to measure the average level of data with eliminating the influence of outliers. Regarding our definition of outliers, we follow the standard approach (e.g., Boxplot) where we calculate the interquartile range (IQR) between the first (Q_1) and third (Q_3) quartile, and all data outside of the range $[Q_1 - 1.5 \cdot IQR, Q_3 + 1.5 \cdot IQR]$ are considered as outliers.

Figure 7 displays our results on total liquidity, total equity, number of insolvent banks and total recovery rate for each of the algorithms RB$_x$ and RIB$_x$, for $x \in \{25, 50, 75, 100\}$, and for each value of default costs $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

Recall that our metrics are expressed as ratios, so values below 1 represent an improvement compared to the original network, and the lower the value, the bigger the improvement.

A detailed discussion on our findings appears at the supplementary material. Overall, our findings imply that debt transfers can effectively improve the well-being of a financial system regarding total liquidity, total equity, number of insolvent banks and total recovery rate, for a wide range of α values; total liquidity improves for $\alpha \leq 0.6$ but each other metric improves for almost all values of α. By comparing the various algorithms, it seems that RIB$_{100}$, which

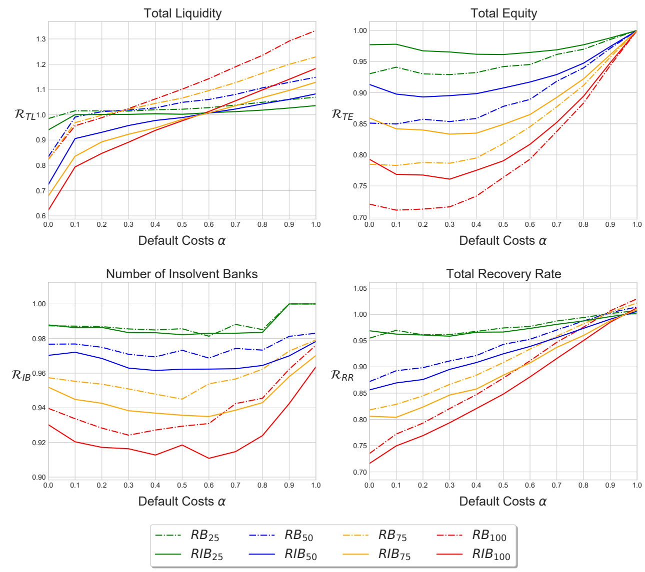

Figure 7: Performance comparison between RB$_x$ and RIB$_x$, for $x \in \{25, 50, 75, 100\}$.

performs all eligible debt transfers of insolvent banks, outperforms the others. Moreover, with the exception of the number of banks in default, the other plots seem to demonstrate an upward trend as α increases which implies that debt transfers have a better effect in systems with high costs of default (α is low), where there is less money flow through the network.

We now consider AIB, a repeated variant of RIB$_{100}$ which was the best performing algorithm among RB$_x$ and RIB$_x$ for $x \in \{25, 50, 75, 100\}$. In the supplementary material we compares the (one-round) RIB$_{100}$ with AIB in terms of total liquidity, total equity, number of insolvent banks and total recovery rate for each value of default costs. Overall, AIB shows a better performance which supports even further the assumption that debt transfers of insolvent banks can improve the well-being of a financial system in several aspects.

5.2 Empirical Analysis of Debt Transfer Games

In this section, we examine debt transfer games in practice. We are interested in observing how fast an initial network N will converge to a "stable" network, i.e., one with no more eligible debt transfers, if banks are allowed to transfer their debt claims strategically. We examine whether such observed outcomes demonstrate improved properties compared to N with respect to total liquidity, total equity, number of insolvent banks and total recovery rate, and also compare them to the best outcomes that emerge in our experiment.

5.2.1 Experimental Setup for Debt Transfer Games.

Network generation. Our choice of parameters regarding generating a set of random networks is shown in Table 1; see the supplementary material for a more detailed discussion.

Game details. We consider the game that emerges when the banks in the aforementioned network structures behave strategically about transferring their debt claims. Each bank/player selects

Table 1: Debt transfer Game Setup

	Parameters
Number of Banks	10
Outgoing Edge Degree	$U(0, 9)$
External Assets	$U(0, 40)$
Liabilities	$U(0, x), x \in \{10, 20, 35\}$
Number of Banks with no External Assets	$U(0, 10)$

between two strategies: transfer all eligible debt claims or do not perform any debt transfers at all. We consider two game variations with respect to how the utility of the players is calculated, namely the individual utility is equal to the equity or the total assets of the corresponding bank. In each case, we consider the game played in rounds. In Round 1, we consider the initial network N and identify banks that have eligible debt transfers. Using the *game_theory* module in *QuantEcon.py*[3], we determine all "stable" outcomes and corresponding networks, where the banks that have been identified in this round are at equilibrium, and randomly pick one of them, call that N_1. If a bank v_i transfers their debt claim from v_j to v_k in N_1 and and edge (v_j, v_k) already existed in N, then its liability is increased by l_{ji} as a result of the debt transfer; note that this might create additional eligible debt transfers. Each following round $i \geq 2$ repeats round $i - 1$ while considering N_{i-1} in place of the initial network. The process stops when the network under consideration at a given round has no additional eligible debt transfers.

Evaluation metrics. We evaluate the performance of the outcomes of the game defined above according to four criteria, namely total liquidity, total equity, number of insolvent banks, and total recovery rate. In the supplementary material we also discuss metrics relevant to the quality of Round 1 "stable" outcomes.

5.2.2 Results. Similarly to the centralized case, in our results, the trimmed mean of corresponding datasets is displayed, where we calculate the mean of the data after discarding outliers; see beginning of Section 5.1.2 for justification and exact definition of outliers.

We run experiments on debt transfer games when banks wish to maximize their total assets and we observed that for a big majority of them (i.e., 7051 out of 11000), the initial network N is identified as the unique "stable" outcome of Round 1. This means that these games do have eligible debt transfers in their initial network, but banks strategically chose not to execute them. This is not surprising as doing a debt transfer implies giving up an incoming debt which directly reduces a bank's total assets. Compensating for that loss in utility through the network would be uncommon, so in most cases a bank would prefer to not transfer its debt claims.

In what follows, we turn our attention to debt transfer games where each bank wants to maximize its equity and the social welfare is defined as the total equity of all banks. Intuitively, banks are more likely to improve their equity by transferring their debt claims, because the loss of the actual incoming payment they are foregoing is always no more than the nominal liability they will avoid.

Our analysis, in the supplementary material, demonstrates that, although the "stable" outcome of the game in the final round is not always strictly better than the initial network, it holds that approximately 75% of all networks are at least as good as the initial network in terms of total equity.

Figure 8 compares the change identified in the total liquidity, the total equity, the number of insolvent banks, and the total recovery rate, between the initial network and the final "stable" outcome of the game; a detailed discussion appears at the supplementary material.

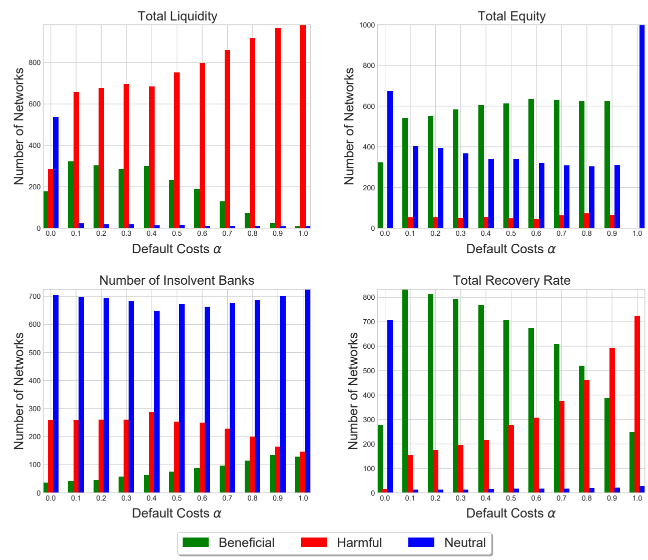

Figure 8: Effect of strategic debt transfers when banks want to maximize their equity.

Recalling the fact that the debt transfer game, for a big majority of the networks considered, terminated at Round 1 implies that results on the quality of "stable" outcomes of Round 1 can offer some intuition with respect to the quality of all "stable" outcomes. Relevant results are discussed in the supplementary material.

Overall, our results support the assumption that debt transfers can be a beneficial operation to both individual banks as well as the entire network.

6 CONCLUSION

We considered the impact of debt transfers on financial networks. Our results indicate that it is computationally hard to identify the optimal collection of debt transfers to maximize systemic liquidity, or to achieve similar objectives related to the well-being of financial networks. Furthermore, we studied the strategic games arising from such operations and focused on the existence, computation, and quality of Nash equilibria. Our theoretical investigations were complemented with experimental study on synthetic networks, both for the centralized and the distributed setting.

Our work leaves some interesting open questions for further study. Regarding Nash equilibria, do they always exist in debt transfer games where banks wish to maximize their total assets and $\alpha \in \{0, 1\}$? Similarly, when banks wish to maximize their equity, do equilibria always exist when $\alpha \in [0, 1)$? Focusing on approximation algorithms is another interesting direction.

[3]https://github.com/QuantEcon/QuantEcon.py

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