A Nash-Bargaining-Based Mechanism for One-Sided Matching Markets with Endowments and Dichotomous Utilities

Extended Abstract

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ABSTRACT

Mechanisms based on maximizing Nash Social Welfare (NSW) have proven to be fair and efficient for a wide variety of fair division problems. We study the fractional allocations maximizing NSW, i.e., a Nash-bargaining-based mechanism, for one-sided matching markets with endowments, under dichotomous utilities, and show that they are the solutions of a rational convex program (RCP). Moreover, we provide a simple combinatorial polynomial time algorithm to maximize NSW by identifying the Nash bargaining points with the equilibrium of a novel type of market, the *variable-budget market model*. Lastly, we show that maximizing NSW is strategyproof under the assumption that the agents' disagreement utilities are public knowledge.

KEYWORDS

Nash Bargaining Solution; Rational Convex Program; Dichotomous Utilities

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1 INTRODUCTION

A one-sided matching market is defined by a set of n indivisible goods and a set of n agents with preferences. The objective is to find a matching of each agent to a distinct good that has desirable fairness and efficiency properties. These markets can be classified along two directions: whether the preferences are cardinal or ordinal, and whether agents have initial endowments or not. Cardinal preferences are more expressive than ordinal, and similarly, the setting with endowments is more general than the no-endowments case.

For the ordinal setting, three popular mechanisms are Probabilistic Serial [9], Random Priority [23], and Top Trading Cycles [25]. The first two are with no endowments, and the last one is with endowments. These mechanisms are polynomial-time computable and have their pros and cons, detailed in Section 1.1. For the cardinal setting with no endowments, the famous mechanism is the Hylland-Zeckhauser (HZ) scheme [21], based on competitive equilibrium. The main difficulty in using the HZ scheme in practical applications is that computing even an approximate equilibrium in this model is PPAD-complete [13, 26]. However, under dichotomous preferences, a combinatorial polynomial-time algorithm [26] and a rational convex program (RCP) [20] exist.

In this paper, we consider the more general setting of cardinal preferences with endowments, which has several natural applications beyond the no-endowments case, e.g., allocating students to rooms in a dorm for the next academic year, assuming their current room is their initial endowment; and school choice, when a student's initial endowment is a seat in a school which they already have. The HZ scheme with endowments was studied by Hylland and Zeckhauser; however, their work culminated in an example which inherently does not admit an equilibrium [21]. This led to defining and studying approximation-based mechanisms [17, 20]. However, they are likely to be computationally intractable.

In view of the above-stated difficulties in terms of existence and computational tractability, we explore an alternative solution for a one-sided matching market with endowments, namely using a Nash-bargaining-based approach. The Nash bargaining solution has very desirable properties: it is computationally tractable, Pareto optimal, symmetric, and has been found to be remarkably fair, e.g., [2, 11, 23]. This aspect has been further explored under the name of Nash Social Welfare [6, 12, 14, 15, 18, 19, 22]. In Section 1.3, we define the notion of a *Nash-bargaining based one-sided matching market with initial endowments*, which we abbreviate to *1NB*.

We explore the well-studied case of dichotomous utilities; see Section 1.1 for related work. Optimal allocations to an instance of 1NB are obtained by optimally solving the non-linear convex program (1) (Section 1.4). We ask if 1NB is polynomial-time solvable. A prerequisite for this is that each instance of 1NB should admit a rational equilibrium. We establish this by showing that 1NB under dichotomous utilities admits an RCP. Furthermore, our proof provides valuable insights which lead us to a simple combinatorial strongly polynomial-time algorithm: It turns out that the dual of this convex program has two types of variables, one corresponding to goods and the other corresponding to agents. Our proof of RCP reveals the roles of these variables – as *prices of goods* and *price-offsets for agents*. Additionally, it indicates how the money, m_i of an agent i should be defined and exactly how i's allocation needs to be paid for.

Using these insights, we give a novel market whose equilibrium captures an optimal solution to program (1). We call it the *variable-budget market model*; it can be viewed as a modification of the

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linear Fisher model. We next give a simple combinatorial strongly polynomial-time *mechanism* for computing an equilibrium for this model. It turns out that m_i is a function of the eventual utility, v_i of agent i. Our mechanism iteratively updates v_i, and as a result, it also keeps updating m_i, hence the name of the model. Finally, by exploiting the combinatorial structure underlying this mechanism, we manage to show it is also strategyproof under the assumption that agents' disagreement utilities are public knowledge.

1.1 Related Results

Matching markets have found applications in various multi-agent settings, see e.g., [1, 3, 4, 8, 16].

We first state the properties of the mechanisms for one-sided matching markets listed in the Introduction. Random Priority [23] is strategyproof though not efficient or envy-free; Probabilistic Serial [9] is efficient and envy-free but not strategyproof; and Top Trading Cycles [25] is efficient, strategyproof and core-stable.

The study of the dichotomous case of matching markets was initiated by Bogomolnaia and Moulin [10]. They studied a two-sided matching market and they called it an "important special case of the bilateral matching problem." Using the Gallai-Edmonds decomposition of a bipartite graph, they gave a mechanism that is Pareto optimal and group strategyproof. They also gave a number of applications of their setting, some of which are natural applications of one-sided markets as well, e.g., housemates distributing rooms, having different features, in a house. As in the HZ scheme, their mechanism also outputs a doubly-stochastic matrix whose entries represent probability shares of allocations. However, they give another interesting interpretation of this matrix. They say, "Time sharing is the simplest way to deal fairly with indivisibilities of matching markets: think of a set of workers sharing their time among a set of employers." Roth, Sönmez and Ünver [24] extended these results to general graph matching under dichotomous utilities; this setting is applicable to the kidney exchange marketplace.

The paper [2] defines the notion of a random partial improvement mechanism for a one-sided matching market. This mechanism truthfully elicits the cardinal preferences of the agents and outputs a distribution over matchings that approximates every agent's utility in the Nash bargaining solution. For dichotomous preferences, [7] showed that a Nash social welfare maximizing allocation can be computed in polynomial-time and [5] studied truthful mechanisms.

1.2 One-Sided Matching Markets

A *one-sided matching market* consists of two types of entities, say agents and goods, with only one side having preferences over the other, i.e., agents over goods. Let $A = \{1, 2, \ldots n\}$ be a set of n agents and $G = \{1, 2, \ldots, n\}$ be a set of n indivisible goods. The goal is to allocate exactly one good to each agent ex-post (final integral allocation).

Goods are rendered divisible by assuming that there is one unit of probability share of each good, and utilities u_{ij}'s are defined as the utility of agent i for the entire unit of good j. In the case of dichotomous utilities, each $u_{ij} \in \{0, 1\}$. Let x_{ij} be the allocation of probability share that agent i receives of good j. Then, $\sum_j u_{ij}x_{ij}$ is the *expected utility* accrued by agent i.

In the final ex-ante allocation, the total probability share allocated to each agent is one unit, i.e., the entire allocation must form a *fractional perfect matching* in the complete bipartite graph over vertex sets A and G (i.e., $\sum_{j \in G} x_{ij} = 1, \forall i \in A;\ x_{ij} \geq 0$). Clearly, the ex-ante allocation can be viewed as a doubly stochastic matrix. The Birkhoff-von Neumann procedure then extracts a random underlying perfect matching in such a way that the expected utility accrued to each agent from the integral perfect matching (i.e., $\sum_{j \in G} x_{ij} = 1, \forall i \in A;\ x_{ij} \in \{0, 1\}$) is the same as from the fractional perfect matching. Since ex-ante Pareto optimality implies ex-post Pareto optimality, the final integral allocation will also be Pareto optimal if its corresponding ex-ante allocation is Pareto optimal.

1.3 The Model *1NB*

In this section, we define the model *Nash-bargaining-based one-sided matching market with initial endowments*, which we abbreviate to *1NB*. We are given a fractional perfect matching x_E which specifies the initial endowments of all agents, each agent getting a total of one unit of goods. Clearly x_E and the utility functions of all agents define the utility accrued by each agent from her initial endowment. We will take this to be agent i's disagreement point c_i and will interpret the problem as a Nash bargaining problem.

The feasible set $\mathcal{N}$ is defined as follows. Let x be a fractional perfect matching over the agents A and goods G and let v_x be an n-dimensional vector whose components are the utilities derived by the agents under the allocation given by x. Then $\mathcal{N}$ is the set of all v_x corresponding to all fractional perfect matchings x.

1.4 RCP for 1NB under Dichotomous Utilities

In this section we will show that program (1) for the model *1NB* is a rational convex program for the case of dichotomous utilities. This will establish the useful property that the model always admits an optimal solution using rational numbers — a pre-requisite for seeking a combinatorial, efficient algorithm. In addition, it will provide important insights into the nature of the dual variables p_j and q_i; these will help in defining a market model whose equilibria correspond to the optimal solutions of (1).

$$\max \quad \sum_{i \in A} \log(v_i - c_i) \tag{1a}$$

$$\text{s.t.} \quad v_i = \sum_j u_{ij} x_{ij} \quad \forall i \in A, \tag{1b}$$

$$\sum_j x_{ij} \leq 1 \quad \forall i \in A, \tag{1c}$$

$$\sum_i x_{ij} \leq 1 \quad \forall j \in G, \tag{1d}$$

$$x_{ij} \geq 0 \quad \forall i \in A, \forall j \in G \tag{1e}$$

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