Single-Peaked Jump Schelling Games

Extended Abstract

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ABSTRACT

We contribute to the recent endeavor of investigating residential segregation models with realistic agent behavior by studying Jump Schelling Games with agents having a single-peaked utility function. In such games, there are empty nodes in the graph and agents can strategically jump to such nodes to improve their utility.

KEYWORDS

Residential Segregation; Non-cooperative Games; Nash Equilibria; Price of Anarchy; Game Dynamics; Computational Hardness

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1 INTRODUCTION

Residential segregation [20], i.e., the emergence of regions in metropolitan areas that are homogeneous in terms of ethnicity or socio-economic status of its inhabitants, has been widely studied by social scientists, mathematicians and, recently, also by computer scientists. Segregation has many negative consequences for the inhabitants of a city, for example, it negatively impacts their health [1].

The causes of segregation are complex and range from discriminatory laws to individual action. In Schelling's classical agent-based model for residential segregation [17, 18] two types of agents, placed on a path or a grid, act according to the following threshold behavior: agents are *content* with their current position if at least a τ-fraction of neighbors, with $\tau \in (0,1)$, is of their own type. Otherwise, they want to move, either via swapping with another random discontent agent or via jumping, to an empty position.

Schelling's model recently gained traction within Algorithmic Game Theory, Artificial Intelligence, and Multi-Agent Systems [2–4, 7–10, 13, 14]. Most of these papers are in line with the assumptions made by Schelling and incorporate monotone utility functions, i.e., the agents' utility is monotone in the fraction of same-type neighbors. However, recent sociological surveys [19] show that people actually prefer to live in diverse rather than segregated neighborhoods[1]. Based on these observations, different models in which agents prefer integration have been proposed [16, 21, 22].

Very recently Bilò et al. [3] introduced and analyzed the Single-Peaked Swap Schelling Game, where agents have single-peaked utility functions and pairs of agents can swap their locations if this is beneficial for both of them. We now take the natural next step and investigate the Jump Schelling Game, where agents can improve their utility by jumping to empty locations, assuming realistic agents having a single-peaked utility function.

Model. We consider a strategic game played on an undirected, connected graph $G = (V, E)$. A *Single-Peaked Jump Schelling Game* (G, r, b, Λ), called the *game*, is defined by a graph G, a pair of positive integers with $r \geq 1$ and $1 \leq b \leq r$ and a peak $\Lambda \in (0,1)$. There are two types of agents, which we associate with the colors red and blue. We have r red agents and $b \leq r$ blue agents. For an agent i, let $c(i)$ be her color. An agent's *strategy* is her position $v \in V$ on the graph. Each node can only be occupied by at most one agent. The $n = r+b$ strategic agents occupy a strict subset of the nodes in V, i.e., there are $e = |V| - n \geq 1$ *empty nodes.* A *strategy profile* $\sigma \in V^n$ is a vector of n distinct nodes in which the i-th entry $\sigma(i)$ corresponds to the strategy of the i-th agent. For convenience, we use σ^{-1} as a mapping from a node $v \in V$ to the agent occupying v or $\ominus$ if v is empty. The set of empty nodes is $\emptyset(\sigma) = \{v \in V \mid \sigma^{-1}(v) = \ominus\}$.

For an agent i, we define $C_i(\sigma) = \{v \in V \setminus \emptyset(\sigma) \mid c(\sigma^{-1}(v)) = c(i)\}$ as the set of nodes occupied by agents of the same color in σ. The *closed neighborhood* of an agent i in a strategy profile σ is $N[i,\sigma] = \{\sigma(i)\} \cup \{v \in V \setminus \emptyset(\sigma) \mid \{v, \sigma(i)\} \in E\}$. The agents care about the fraction $f_i(\sigma)$ of agents of their own color, including themselves, in their closed neighborhood where $f_i(\sigma) = \frac{|N[i,\sigma] \cap C_i(\sigma)|}{|N[i,\sigma]|}$. Observe that we have $f_i(\sigma) > 0$ for any agent i, since $\sigma(i) \in N[i,\sigma]$. Also, we emphasize that our definition of $f_i(\sigma)$ deviates from similar definitions in related work [2, 9, 10, 13]. Similar to Bilò et al. [3],

[1]Respondents (on average 78% white) were asked what they think of "Living in a neighborhood where half of your neighbors were blacks?". A clear majority, e.g. 82% in 2018, responded "strongly favor", "favor" or "neither favor nor oppose".

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the key idea of our definition is that agents contribute to the diversity of their neighborhood, i.e., they actively strive for integration. We think that this best captures the single-peaked setting.

The *utility* of an agent i is $U_i(\sigma) = p(f_i(\sigma))$, with p being an arbitrary single-peaked function with peak $\Lambda \in (0,1)$ and the following properties: (1) $p(0) = 0$ and $p(x)$ is strictly monotonically increasing on $[0,\Lambda]$, (2) for all $x \in [\Lambda, 1]$ it holds that $p(x) = p(\frac{\Lambda(1-x)}{1-\Lambda})$. W.l.o.g., we further assume that $p(\Lambda) = 1$.

An agent can change her strategy by performing a *jump*, i.e., to choose an empty node $v \in \emptyset(\sigma)$ as new location. We denote the resulting strategy profile after a jump of agent i to a node v as σ_{iv}. A jump is *improving*, if $U_i(\sigma) < U_i(\sigma_{iv})$. A strategy profile σ is a (pure) *Nash Equilibrium* (NE) if and only if there are no improving jumps, i.e., for all agents i and nodes $v \in \emptyset(\sigma)$, we have $U_i(\sigma) \geq U_i(\sigma_{iv})$.

A measure to quantify the amount of segregation in a strategy profile σ is the *degree of integration* (DoI), which counts the number of non-segregated agents, hence $\text{DoI}(\sigma) = |\{i \mid f_i(\sigma) < 1\}|$. For a game, let σ^* be a strategy profile that maximizes the DoI and let NE be its set of Nash Equilibria. We evaluate the impact of the agents' selfishness on the social welfare by studying the *Price of Anarchy* (PoA) and the *Price of Stability* (PoS) with respect to the DoI.

A game has the *finite improvement property* (FIP) if and only if, starting from any strategy profile σ, the game will always reach a NE in a finite number of steps. The FIP does not hold if there is a cycle of strategy profiles $\sigma^0, \sigma^1, \ldots, \sigma^k = \sigma^0$, such that for any $k' < k$, there is an agent i and empty node $v \in \emptyset(\sigma^{k'})$ with $\sigma^{k'+1} = \sigma^{k'}_{iv}$ and $U_i(\sigma^{k'}) < U_i(\sigma^{k'+1})$. These cycles are known as *improving response cycles* (IRCs).

Related Work. Game-theoretic models for residential segregation were first studied by [9, 10]. There, agents have a monotone utility function. Agarwal et al. [2] consider a simplified model using a monotone threshold-based utility function with $\tau = 1$ and they introduce the DoI as social welfare measure. Kreisel et al. [15] show that deciding the existence of NE in the swap version as well as in the jump version of the simplified model is NP-hard. Bilò et al. [4] strengthened the PoA results for the swap version w.r.t. the utilitarian social welfare and investigated the model on specific graph classes. Other variants are studied in [8, 13, 14]. Bullinger et al. [7] measure social welfare via the number of agents with non-zero utility, they prove hardness results for computing the social optimum and discuss other solution concepts.

Most related is the recent work by [3], which studies the swap-version of our model. They find that equilibria are not guaranteed to exist in general, but they do exist for $\Lambda = \frac{1}{2}$ on bipartite graphs and for $\Lambda \leq \frac{1}{2}$ on almost regular graphs. The latter is shown via an ordinal potential function, i.e., convergence of IRDs is guaranteed. For the PoA they prove an upper bound of $\min\{\Delta(G), \frac{n}{b+1}\}$, where $\Delta(G)$ is the maximum degree in G, and give almost tight lower bounds for bipartite graphs and regular graphs. Also, they lower bound the PoS by $\Omega(\sqrt{n\Lambda})$ and give constant bounds on bipartite and almost regular graphs. Note that due to the existence of empty nodes in our model, our results cannot be directly compared.

Also related are hedonic diversity games [5, 6, 12] where selfish agents form coalitions and the utility of an agent only depends on the type distribution of her coalition. For such games, single-peaked utility functions yield favorable game-theoretic properties.

Our Contribution. We investigate Jump Schelling Games with agents having a single-peaked utility function. Such functions better reflect recent sociological poll results on real-world agent behavior [19]. Moreover, this is also interesting from a technical point of view since it yields insights into the properties of Schelling-type systems under different preconditions.

Regarding NE existence, we provide a collection of positive and negative results. On the negative side, we show that NE are not guaranteed to exist on the simplest possible topologies, i.e., on paths and rings with single-peaked utilities with $\Lambda \geq \frac{1}{2}$. This is in contrast to the version with monotone utilities where for the case of rings NE always exist. On the positive side, we give various conditions that enable NE existence, e.g., they exist if the underlying graph has a sufficiently large independent set, or if it has sufficiently many degree 1 nodes. For game dynamics, the situation is worse. We show that even on regular graphs IRCs exist independently of the position of the peak. Moreover, this even holds for the special case with $\Lambda = \frac{1}{2}$ and only a single empty node. These results for $\Lambda \leq \frac{1}{2}$ also represent a marked contrast to the swap version, where convergence is guaranteed on almost regular graphs.

With regard to the quality of the equilibria, we focus on the DoI as social cost function. For the PoA w.r.t. the DoI, we establish that the technique from [3] for deriving a PoA upper bound can be adapted to also work in our setting. This yields the same PoA upper bound of $\min\{\Delta(G), n/(b+1)\}$. Subsequently, we give almost matching PoA lower bounds and we prove that also the lower bounds for the PoS almost match this high upper bound. On the positive side, we show that on graphs with a sufficiently large independent set, the PoS depends on the ratio of the largest and the smallest node degree, which implies a PoS of 1 on regular graphs that also holds for rings with a single empty node.

Last but not least, we consider complexity aspects of our model. Analogously to previous work on the Jump Schelling Game with monotone utilities and to work on the Single-Peaked Swap Schelling Games, we focus on the hardness of computing a strategy profile with a high DoI. Using a novel technique relying on the Max SAT problem, we show that this problem is NP-complete, improving on an earlier result by [2]. Moreover, as a novel conceptual contribution, we investigate the hardness of finding an equilibrium state via improving response dynamics. As one of our main results, we show that this problem is NP-hard. So far, researchers have studied the complexity of deciding the existence of an equilibrium for a given instance of a Schelling Game. We depart from this, since even if it can be decided efficiently that for some instance an equilibrium exists, guiding the agents towards this equilibrium from a given initial state is complicated, since this would involve a potentially very complex centrally coordinated relocation of many agents in a single step. In contrast, reaching an equilibrium via a sequence of improving moves is much easier to coordinate, since in every step the respective move can be recommended and, since this is an improving move, the agents will follow this advice.

Overall we find that making the model more realistic by employing single-peaked utilities entails a significantly different behavior of the model compared to the variant with monotone utilities but also compared to Single-Peaked Swap Schelling Games.

See Friedrich et al. [11] for the full version of this paper.

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