Allocating Contiguous Blocks of Indivisible Chores Fairly: Revisited

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ABSTRACT

Resource allocation is a fundamental problem in multi-agent systems, with two key factors to consider: fairness and efficiency. The concept of the "price of fairness" helps in the understanding of efficiency loss under fairness constraints. Among the diverse resource allocation settings, cake cutting stands out as a prominent model. Recently, Höhne and van Stee [Inf. Comput., 2021] examined a variation of this model in which the cake represents indivisible chores, with each agent requiring a connected piece of the chores. Höhne and van Stee provided upper and lower bounds on the price of fairness when fairness is measured by envy-freeness and proportionality. However, in the case of indivisible items, achieving envy-free and proportional allocations is difficult, rendering these bounds insufficient for a comprehensive understanding of the true trade-off between fairness and efficiency. In this paper, we revisit the same problem and consider fairness notions that are satisfiable, including proportionality up to one item, and maximin share fairness. By presenting tight bounds on the price of fairness with respect to these notions, we complete the picture of fairness and efficiency trade-off.

KEYWORDS

Price of fairness; Indivisible chores; MMS; PROP1

ACM Reference Format:
Ankang Sun and Bo Li. 2024. Allocating Contiguous Blocks of Indivisible Chores Fairly: Revisited. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 9 pages.

1 INTRODUCTION

Resource allocation is a fundamental problem in various multi-agent systems, where two crucial but orthogonal factors come into play: fairness and efficiency. Traditionally, research in resource allocation has predominantly addressed either efficiency or fairness separately [6, 8, 16], and seminal works by Bertsimas et al. [14] and Caragiannis et al. [20] introduced the concept of "price of fairness" to study the impact of fairness on the efficiency of allocations. Since then, a significant body of research has emerged, focusing on bounding the price of fairness in diverse resource allocation settings.

Cake cutting serves as a prominent model and is capable of capturing a wide range of real-life scenarios in a simple yet powerful framework [18, 24, 25, 36]. The cake-cutting problem involves dividing a heterogeneous resource, represented by an interval $[0, 1]$, among multiple agents with different preferences and valuations. The goal is to find a division that satisfies some predetermined objectives, including but not limited to fairness and efficiency. While the canonical cake allocation has been well-established and extensively studied, recent research has started to explore more about its variants, acting as metaphors for practical resource allocation problems. In particular, [28, 30, 31, 37] studied the discrete version of the cake cutting problem, where the interval is replaced by a path of vertices, with each vertex representing an indivisible item, and each agent is required to receive a contiguous block of items (i.e., connectivity constraints). This discrete version is particularly used to capture resource allocation problems where the underlying resource has a temporal or spatial structure, such as allocating conference sessions to different organizers, assigning workload to bin workers in a local district and allocating network rail maintenance activities to maintenance employees. In these allocation problems, each agent prefers to receive a contiguous block of items.

In the model of allocating contiguous blocks of indivisible items, the trade-off between fairness and efficiency has already drawn attention of researchers. Suksompong [37] and Höhne and van Stee [28], respectively, bounded the price of fairness for goods and chores of the discrete cake cutting problem. The fairness criteria therein are envy-freeness (EF), proportionality (PROP) and equitability (EQ) – three gold-standard fairness notions.[1] For indivisible items, it is widely known that EF, PROP and EQ are very hard to satisfy, resulting in the corresponding bounds of price of fairness falling short in providing a comprehensive understanding of the true trade-off between fairness and efficiency: the impact of enforcing fairness on the efficiency loss was not considered for the instances which do not admit EF or PROP or EQ allocations. Similar limitations were observed in the earlier work of [20] which studied the price of fairness of indivisible items without the connectivity constraints. To overcome these limitations, Bei et al. [13] and Barman et al. [11] took a different approach by shifting to the fairness notions that are always satisfiable.

In line with [11, 13], Sun and Li [40] considered the allocation of indivisible goods with connectivity constraints, the same model as [37], and provided the bounds of the price of fairness regarding PROP1 and MMS – two popular relaxed notions of proportionality that can always be satisfied. The established price of fairness in [40] also indicates the limitation of studying non-satisfiable fairness

[1] Intuitively, an allocation is EF is every agent prefers her own items than any other agent's; is PROP if every agent gets at least $\frac{1}{n}$ of her total utility for all items, where n is the number of agents; is EQ if agents' utilities are at the same level.

Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), N. Alechina, V. Dignum, M. Dastani, J.S. Sichman (eds.), May 6 – 10, 2024, Auckland, New Zealand.

criteria. However, the issue remains unresolved for the problem of chores, which motivates our work. Although, at first glance, allocating chores looks like a symmetric or dual problem of the allocation of goods, it has been observed that allocating goods or chores are not mirror images of one another. Thus, in this paper, we revisit fairness and efficiency trade-off in the model of allocating contiguous blocks of indivisible chores (that in [28]) by considering satisfiable fairness notions. Our objective is to complete the picture of efficiency loss under fairness constraints by establishing the price of fairness regarding the fairness criteria whose existence is guaranteed in all instances of allocating contiguous blocks of indivisible chores.

1.1 Contributions

In this paper, we quantify the efficiency loss under fairness constraints by establishing the corresponding price of fairness ratios. The underlying fairness notions are *proportionality up to one item* (PROP1) and *maximin share* fairness (MMS). The efficiency of an allocation is assessed through two welfare functions; *utilitarian* welfare is the summation of individuals' value, and *egalitarian* welfare is the value of the worst-off agent. We first consider the general case of $n \geq 3$. In terms of egalitarian welfare, the price of MMS is $\frac{n}{2}$, while the price of PROP1 is $\frac{n}{2}$ for $n \geq 4$ and is 2 for $n = 3$. When efficiency is measured by utilitarian welfare, the price of MMS is at most $3n$ and at least $\frac{n+2}{4}$, asymptotically tight $\Theta(n)$. For the notion of PROP1, we also establish the asymptotically tight result $\Theta(n)$. These results are summarized in Table 1, and for the ease of comparison, we also provide the known PoF results regarding EF and PROP in [28]. Additionally, we explore the model with two agents and demonstrate, in Section 5, that the prices of MMS and of PROP1 are two and one regarding utilitarian welfare and egalitarian welfare, respectively.

When comparing the results to those in [28], we have two interesting observations. On the one hand, if one relaxes the underlying fairness notion from EF to PROP1 and MMS, the price of fairness decreases from infinity to $\Theta(n)$, which confirms the intuition that the weaker the fairness notion is, the less efficiency would be sacrificed. On the other hand, the price of fairness of PROP, together with that of MMS and PROP1, seems to be counter-intuitive. Specifically, we show that prices of MMS and of PROP1 are $\frac{n}{2}$, way larger than the price of PROP. Note that any PROP allocation is also MMS and PROP1, and hence, the price of PROP1 should be no greater than the price of PROP and of MMS. This counter-intuitive result, as we have discussed, arises from neglecting instances in which no PROP allocation exists when studying the price of PROP. For these challenging instances, a significant portion of efficiency is sacrificed even for allocations that satisfy fairness constraints weaker than PROP.

From the technique perspective, studying the price of fairness regarding relaxed fairness notions is not easier than that of PROP or EF, and even brings more challenges. For example, if PROP allocations exist, then the price of PROP with respect to egalitarian welfare is trivially one as the egalitarian welfare-maximizing allocation would also be PROP. Nonetheless, this argument does not extend to the concepts of MMS and PROP1.

We remark that all upper bounds, in this work, are proven through a constructive approach. For each fairness notion, we utilize the idea of moving-knife and propose an algorithm with a tunable parameter for the purpose of controlling individuals' value. Then, we characterize the parameter domain, with which implementing the proposed algorithm can output allocations satisfying the underlying fairness notion. For utilitarian welfare, by choosing the threshold parameter properly, our parametric algorithm can, in polynomial time, return fair solutions achieving the asymptotically tight upper bound. For egalitarian welfare, if we allow an oracle on computing egalitarian welfare-maximizing allocations, then in most cases (when $n \geq 4$ for MMS and $n \geq 8$ for PROP1), the proposed algorithms can efficiently return fair solutions with the best possible worst-case efficiency guarantee. The remaining situations are proved by carefully reallocating chores upon the allocation returned by the proposed algorithm.

1.2 Other Related Works

Our work is closely related to the rich body of literature on cake cutting, where a divisible resource denoted by the real interval $[0, 1]$ is allocated to a set of agents. For cake cutting problems, an envy-free and proportional allocation always exists [18], and a recent breakthrough paper [8] proved that such an allocation can be found in finite steps. The follow-up work [23] solved this problem for the envy-free allocation of a divisible chore. Su [25] considered the constrained version of this problem, where every agent is required to receive a contiguous piece of the cake, and the resulting price of fairness is analyzed in [5].

However, when the items become indivisible, the problem becomes different and the aforementioned techniques cannot be applied any more, which is the focus of the current work. Without any constraints, approximate envy-freeness and proportionality are guaranteed to be satisfiable [32, 34, 35]. When the allocations are required to satisfy extra constraints, the problem is trickier and the readers can refer to the survey paper [38] for a detailed introduction. One of the most and natural constraints is *connectivity*, where the items are assumed to be distributed on a graph [15, 17] and each agent should receive a connected subgraph. The paths as we considered in this study represent a significant special case that yields some interesting positive results.

Besides the price of fairness, there is another line of research investigating if efficiency and fairness can be satisfied simultaneously, such as the compatibility between Pareto optimality and approximate envy-freeness/proportionality; see, e.g.,[3, 12, 21, 27].

2 PRELIMINARIES

In the model of allocating contiguous blocks of indivisible chores, there is a set $E = \{e_1, \ldots, e_m\}$ of indivisible chores located on a path, and throughout the paper, e_j is at the left of e_{j+1} for all $j \leq m - 1$. Let $N = \{1, \ldots, n\}$ denote the set of n agents, and each agent requires a contiguous block, i.e., connectivity constraint. Both an empty set and a singleton are regarded as connected. We refer to subsets of items as *bundles*, and, moreover, let C be the set of contiguous bundles. Each agent i has a *disutility* or non-positive valuation function $v_i(\cdot) : C \to \mathbb{R}_{\leq 0}$. Similar to existing works on allocations with connectivity constraints [15, 28, 30, 37], for

General n	EF	PROP	MMS	PROP1	
PoF	∞	n	$\Theta(n)$ (Theorems 12 and 18)		Utilitarian
	∞	1	$\frac{n}{2}$ (Theorem 11)	$\frac{n}{2}$ for $n \neq 3$; 2 for $n = 3$ (Theorem 17)	Egalitarian

Table 1: The price of fairness regarding EF, PROP, MMS, PROP1. The ratios for EF and PROP are proved in [28].

any agent i, valuation v_i is assumed to be *additive*, that is, $v_i(C) = \sum_{e \in C} v_i(\{e\})$ for all $C \in \mathcal{C}$, and *normalized*, that is, $v_i(E) = -1$. Throughout the paper, denote by $I = \langle N, E, \{v_i\}_{i=1}^n \rangle$ an instance; by $L(k) = \{e_1, \ldots, e_k\}$; by $R(k) = \{e_k, \ldots, e_m\}$; by $[k] = \{1, \ldots, k\}$ for all $k \in \mathbb{N}^+$. For any agent $i \in [n]$ and chore $e \in E$, instead of $v_i(\{e\})$, we write $v_i(e)$.

A *feasible* allocation $\mathbf{A} := (A_1, \ldots, A_n)$ is an n-partition of E where every bundle must be contiguous, i.e., for any $i \neq j$, $A_i \cap A_j = \emptyset$, $\cup_{i \in N} A_i = E$ and $A_i \in \mathcal{C}$ for all $i \in N$. If not explicitly stated otherwise, thereafter, all allocations and bundles are feasible and contiguous, respectively. For any subset $S \subseteq E$ and $k \in \mathbb{N}^+$, let $\Pi_k(S)$ be the set of k-contiguous partitions of S, and let $|S|$ be the number of items in S.

2.1 Fairness Notions

We below introduce fairness notions of *proportionality* (PROP) and its relaxation. The notion of PROP requires each agent to receive a value at least $-\frac{1}{n}$.

Definition 1 (PROP). *An allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *is said to be* PROP *if for any* $i \in [n]$, $v_i(A_i) \geq -\frac{1}{n}$.

In the context of indivisible chores, PROP allocation does not always exist. A relaxation, so-called *proportional up to one item* (PROP1) is proposed and has been widely studied in various fair division problem [6, 9, 22].

Definition 2 (PROP1). *An allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *is said to be* PROP1 *if for any* $i \in [n]$, *there exists a chore* $e \in A_i$ *such that* $A_i \setminus \{e\} \in \mathcal{C}$ *and* $v_i(A_i \setminus \{e\}) \geq -\frac{1}{n}$.

Another relaxation of PROP is *maximin share* (MMS) fairness [4, 29, 32]. The rationale of maximin share comes from the generalization of the *cut-and-choose* protocol: agent i is asked to partition chores into n contiguous bundles, but she is the last to choose. In the worst-case scenario, agent i receives the least-value bundle for her. The risk-averse strategy for agent i is to cut in a way that maximizes the minimum value of a bundle. This idea brings about the formal definition below.

$$\mathsf{MMS}_i(E, n) = \max_{\mathbf{X} \in \Pi_n(E)} \min_{j \in [n]} v_i(X_j).$$

We say $\{T_k\}_{k=1}^n$ is an $\mathsf{MMS}_i(E, n)$-*defining* partition if $v_i(T_k) \geq \mathsf{MMS}_i(E, n)$ for all $k \in [n]$. The MMS fairness ensures that every agent i receives a value at least $\mathsf{MMS}_i(E, n)$.

Definition 3 (MMS). *For an allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *of instance* $I = \langle N, E, \{v_i\}_{i=1}^n \rangle$, $\mathbf{A}$ *is said to be* MMS *fair if* $v_i(A_i) \geq \mathsf{MMS}_i(E, n)$ *for all* $i \in N$.

For simple notations, throughout the paper, we write MMS_i when the underlying E and n are clear from the context. We remark that in contrast with the NP-hardness of the computation of MMS without connectivity constraints [10, 33], in our setting, MMS value of an agent can be computed in polynomial time.

Lemma 4. *Given an instance* I, *for any agent* i, *the value* MMS_i *can be computed in polynomial time.*

Due to the page limit, missing proofs can be found in the full version.

2.2 Welfare functions and price of Fairness

In this work, we borrow from the welfare economics two canonical social welfare functions, namely, *utilitarian welfare* and *egalitarian welfare*, to measure the efficiency of outcomes.

Definition 5. *Given an allocation* $\mathbf{A} = (A_1, \ldots, A_n)$, *utilitarian welfare and egalitarian welfare functions of* $\mathbf{A}$ *are* $\mathsf{UW}(\mathbf{A}) = \sum_{i \in [n]} v_i(A_i)$ *and* $\mathsf{EW}(\mathbf{A}) = \min_{i \in [n]} v_i(A_i)$, *respectively.*

The last notion to be introduced is *price of fairness* (PoF) that characterizes, in the worst-case scenario, the efficiency loss under a certain fairness constraint. The PoF is the supremum ratio over all instances between maximum welfare of all fair allocations and maximum welfare of all allocations.

Definition 6 (PoF). *The price of fairness with respect to welfare function* W *and fairness criterion* F *is*

$$\mathsf{PoF}(W \mid F) = \sup_{I} \min_{\mathbf{A} \in F(I)} \frac{W(\mathbf{A})}{\mathsf{OPT}_W(I)},$$

where $\mathsf{OPT}_W(I)$ *refers to the maximum* $W(\mathbf{A})$ *among all allocations* $\mathbf{A}$ *of* I; $F(I)$ *refers to the set of allocations satisfying fairness criterion* F.

In the above definition, we apply the following convention: if the maximum welfare of an instance is equal to zero[2], then the price of fairness is defined to be 1. Note that in the above definition of PoF, we pursue the minimum ratio as the welfare is non-positive in the allocations of chores. For a simple presentation, we use OPT_E and OPT_U to refer to $\mathsf{OPT}_{EW}(I)$ and $\mathsf{OPT}_{UW}(I)$, respectively if the instance is clear from the context. The PoF with respect to fairness criterion F is also called *price of* F, i.e., price of PROP1 or price of MMS.

3 PRICES OF MMS FOR GENERAL $n \geq 3$

We start with MMS fairness. To find and compute the desired MMS allocation, we propose a polynomial time algorithm ALG-M(β) (see Algorithm 1) with a parameter β. ALG-M(β) relies on the idea of a moving knife. It involves starting from the leftmost item

[2]In this case, the welfare-maximizing allocation satisfies both fairness notions considered in this paper.

and iteratively identifying the farthest item such that there exists some agent i whose valuation for items ranging from the leftmost item to the current one does not exceed a predetermined threshold $\max\{\mathsf{MMS}_i, \beta\}$ where β is a non-positive real number. The parameter β, as an input, is incorporated to control an individual's value thereby ensuring a certain level of welfare. In the following, we first present the value range of β allowing ALG-M(β) to return an MMS allocation. Then, we choose the proper β to establish the tight PoF ratio regarding both egalitarian welfare and utilitarian welfare.

Algorithm 1 ALG-M(β)

Input: An instance $I = \langle N, E, \{v_i\}_{i=1}^n \rangle$ and a real number β.
Output: An allocation $\mathbf{A} = (A_1, \dots, A_n)$.

1: Initialize $N_0 \leftarrow N$ and $E_0 \leftarrow E$.
2: **while** $|N_0| > 1$ & $E_0 \neq \emptyset$ **do**
3: Denote by $e_L \in E_0$ the left-most item in E_0.
4: **if** there exists an agent $i \in N_0$ such that $v_i(e_L) \geq \max\{\mathsf{MMS}_i, \beta\}$ **then**
5: Let p be the largest index such that there exists an agent i^* with $v_{i^*}(L(p) \cap E_0) \geq \max\{\mathsf{MMS}_{i^*}, \beta\}$. If there is a tie, pick the agent with largest value on $L(p) \cap E_0$.
6: $A_{i^*} \leftarrow L(p) \cap E_0$, $E_0 \leftarrow E_0 \setminus A_{i^*}$, $N_0 \leftarrow N_0 \setminus \{i^*\}$.
7: **else**
8: Let $i^* \in \arg\max_{j \in N_0} v_j(e_L)$, breaking tie arbitrarily.
9: $A_{i^*} \leftarrow \{e_L\}$, $E_0 \leftarrow E_0 \setminus \{e_L\}$, $N_0 \leftarrow N_0 \setminus \{i^*\}$.
10: **end if**
11: **end while**
12: If $E_0 \neq \emptyset$, assign E_0 to the only agent in N_0.
13: **return A**

Lemma 7. *For any $\beta \leq -\frac{2}{n}$,* ALG-M(β) *returns an* MMS *allocation in polynomial time.*

Proof. By Lemma 4, the value MMS_i's can be computed in polynomial time. With known MMS_i, ALG-M(β) allocates all items in $O(m^2 n^2)$ time since the number of agents is reduced by one in each iteration of while-loop, and all remaining items are assigned to the last agent.

Next we prove that the returned allocation $\mathbf{A} = (A_1, \dots, A_n)$ is MMS. Renumber the agents from 1 to n according to the order of receiving bundles in the algorithm, where agent 1 is the first to receive a bundle and agent n is the last. For an agent $i \in [n-1]$, she receives A_i in either Step 6 or 9. If A_i is assigned in Step 6, we have $v_i(A_i) \geq \max\{\mathsf{MMS}_i, \beta\} \geq \mathsf{MMS}_i$. For the latter, since $|A_i| = 1$, then $v_i(A_i) \geq \mathsf{MMS}_i$ holds. Thus, agents $[n-1]$ satisfy MMS fairness, and the remaining is to show $v_n(A_n) \geq \mathsf{MMS}_n$.

If $v_n(A_n) = 0$, then we are done. As for the case where $v_n(A_n) \neq 0$, we split the proof into two cases.

Case 1: $\mathsf{MMS}_n \geq \beta$. Let $\mathbf{S} = (S_1, \dots, S_n)$ be an MMS_n-defining partition and for p, q with $p < q$, bundle S_p is on the left of S_q. By the order of agents, for any pair of i, j with $i < j$, bundle A_i is on the left of A_j . Then we prove the following claim.

Claim 8. *For any $1 \leq k \leq n-1$, $\bigcup_{j=1}^k S_j \subseteq \bigcup_{j=1}^k A_j$.*

Claim 8 implies that $A_n \subseteq S_n$, and thus $v_n(A_n) \geq v_n(S_n) \geq \mathsf{MMS}_n$, which completes the proof for the case $\mathsf{MMS}_n \geq \beta$.

Case 2: $\mathsf{MMS}_n < \beta$. Under this case, the threshold value in Step 4 becomes β; note that $\beta \leq -\frac{2}{n}$. Due to the ordering of agents, for any $j \in [n-1]$, bundle A_{j+1} is on the right of A_j, and moreover, $A_j \cup A_{j+1} \in C$. We then upper bound agent n's value on any two connected bundles.

Claim 9. *For any $j \in [n-1]$, $v_n(A_j \cup A_{j+1}) < \beta$ holds.*

Then, we show $v_n(A_n) \geq -\frac{1}{n}$ by elementary counting. On the one hand, if $\frac{n-1}{2} \in \mathbb{N}^+$, as $v_n(A_j \cup A_{j+1}) < \beta \leq -\frac{2}{n}$ for all $j \in [n-2]$, it holds that $\sum_{j \in [n-1]} v_n(A_j) < -\frac{n-1}{n}$, implying $v_n(A_n) > -\frac{1}{n} \geq \mathsf{MMS}_n$ due to normalized valuations. On the other hand, if $\frac{n-1}{2} \notin \mathbb{N}^+$, we sum up j from 1 to $n-2$ and have an upper bound of $\sum_{j=1}^{n-2} v_n(A_j) < -\frac{n-2}{n}$. Accordingly, due to normalized valuations, $v_n(A_n) > -\frac{2}{n} \geq \beta > \mathsf{MMS}_n$ holds, and therefore, agent n is satisfied with MMS fairness. □

The above proof also implies that agent n's value is at least β for all $\beta \leq -\frac{2}{n}$. This bound will be used later for characterizing the PoF ratio regarding utilitarian welfare.

Corollary 10. *For any $\beta \leq -\frac{2}{n}$, agent n (the last agent to receive a bundle), in the allocation returned by* ALG-M(β)*, has a value at least β.*

3.1 On egalitarian welfare

In this section, we are concerned with egalitarian welfare, and prove that the tight ratio on the price of MMS is $\frac{n}{2}$. We will investigate the allocation returned by ALG-M($-\frac{1}{2}$) and show that such an allocation is either the desired MMS allocation or a step-stone for finding the desired one.

Theorem 11. *For egalitarian welfare and MMS fairness, the price of fairness is $\frac{n}{2}$.*

Proof. We begin with the upper bound. Due to normalized valuations, if $\mathsf{OPT}_E \leq -\frac{2}{n}$, then any MMS allocation achieves the PoF ratio of $\frac{n}{2}$. Moreover, if $\mathsf{OPT}_E \geq -\frac{1}{n}$, then egalitarian welfare-maximizing allocation is MMS. Thus, we can further assume $-\frac{2}{n} < \mathsf{OPT}_E < -\frac{1}{n}$.

For $n \geq 4$, the assumption becomes $-\frac{1}{2} \leq -\frac{2}{n} < \mathsf{OPT}_E < -\frac{1}{n}$. Hence, it suffices to show that there exists an MMS allocation with egalitarian welfare at least $-\frac{1}{2}$. Denote by $\mathbf{A} = (A_1, \dots, A_n)$ the allocation returned by ALG-M($-\frac{1}{2}$). Without loss of generality, agents are renumbered by the order of receiving bundles in the algorithm, that is, agent 1 is the first to receive a bundle and agent n is the last. By Lemma 7, allocation $\mathbf{A}$ is MMS, and thus, if $\mathsf{EW}(\mathbf{A}) \geq -\frac{1}{2}$, the statement is proved. Below we discuss the situation where $\mathsf{EW}(\mathbf{A}) < -\frac{1}{2}$.

Let agent k be the one such that $v_k(A_k) \leq v_i(A_i)$ for all $i \in [n]$, and hence $v_k(A_k) = \mathsf{EW}(\mathbf{A}) < -\frac{1}{2}$. According to Step 4, bundle A_k is allocated in Step 9 and moreover $|A_k| = 1$, which further implies $\mathsf{MMS}_k = v_k(A_k) < -\frac{1}{2}$. We claim that every other agent receives a value at least $-\frac{1}{2}$; that is, for any $j \neq k$, $v_j(A_j) \geq -\frac{1}{2}$ holds. For any $j < k$, if $v_j(A_j) < -\frac{1}{2}$, then A_j is allocated in Step 9 and moreover $v_k(A_j) \leq v_j(A_j) < -\frac{1}{2}$. Then, we have $-1 > v_k(A_k \cup A_j) \geq v_k(E) = 1$, a contradiction. If there exists some agent

$j > k$ with $v_j(A_j) < -\frac{1}{2}$. Note that $v_j(A_k) < -\frac{1}{2}$; otherwise, A_k is not allocated to agent k. Similarly, we have $-1 > v_j(A_k \cup A_j) \geq v_j(E) = -1$, a contradiction.

As $\mathsf{OPT}_E > -\frac{1}{2}$, there exists an agent p^* (with $p^* < k$) who receives bundle A_k in the egalitarian welfare-maximizing allocation and has a value $v_{p^*}(A_k) > -\frac{1}{2}$. Construct another allocation $\mathbf{B}$ where $B_{p^*} = A_k$, $B_k = A_{p^*}$, and $B_j = A_j$ for all $j \neq p^*, k$. Allocation $\mathbf{B}$ is clearly contiguous and moreover, for any agent $j \neq p^*, k$, it holds that $v_j(B_j) \geq -\frac{1}{2}$ and $v_j(B_j) \geq \mathsf{MMS}_j$. As for agents k and p^*, both of them receive a value at least $-\frac{1}{2}$. Moreover, agent k satisfies MMS as $v_k(B_k) > -\frac{1}{2} \geq \mathsf{MMS}_k$ and agent p^* is also happy regarding MMS as $|B_{i^*}| = 1$. Therefore, allocation $\mathbf{B}$ is MMS and has an egalitarian welfare at least $-\frac{1}{2}$, which implies that the price of MMS regarding egalitarian welfare is at most $\frac{n}{2}$ when $n \geq 4$.

When $n = 3$, if $\mathsf{OPT}_E > -\frac{1}{2}$, then by similar arguments as the case of $n \geq 4$, one can prove the price of MMS is at most $\frac{3}{2}$. As for the case of $\mathsf{OPT}_E \leq -\frac{1}{2}$, there exists an item $\tilde{e}$ such that $v_i(\tilde{e}) \leq -\frac{1}{2}$ holds for all $i \in [3]$, implying $\mathsf{MMS}_i \leq -\frac{1}{2}$ for all $i \in [3]$. As a consequence, the egalitarian welfare-maximizing allocation is also MMS so that the price of MMS in this case is equal to one. Up to here, we show that price of MMS regarding egalitarian welfare is at most $\frac{n}{2}$ for $n \geq 3$.

As for the lower bound, let us consider an instance with $n \geq 3$ agents and a set $E = \{e_1, \ldots, e_{n+2}\}$ of $n + 2$ chores. The valuations are shown in the following table, where $\epsilon > 0$ is arbitrarily small.

Chores	e_1	e_2	e_3	e_4&e_5	e_6	$\cdots$	e_{n+2}
$v_1(\cdot)$	$-\frac{1}{n}$	$-\epsilon$	$-\epsilon$	$-\frac{1}{n}+\epsilon$	$-\frac{1}{n}$	$\cdots$	$-\frac{1}{n}$
$v_i(\cdot)$ for $i \geq 2$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\cdots$	0

Table 2: The lower bound instance in Theorem 11

In an egalitarian welfare maximizing allocation $\mathbf{O}$, agent 1 receives bundle $O_1 = \{e_1, e_2, e_3\}$ and has a value $v_1(O_1) = -\frac{1}{n} - 2\epsilon$. However, one can verify that $\mathsf{MMS}_1 = -\frac{1}{n} - \epsilon > v_1(O_1)$, and thus, allocation $\mathbf{O}$ is not MMS. Note that in any MMS allocation, items e_1 and e_3 cannot be assigned to agent 1 at the same time. Accordingly, the egalitarian welfare of an MMS allocation is at most $-\frac{1}{2}$, and thus the price of MMS with respect to egalitarian welfare is at least

$$\mathsf{PoF}(\mathsf{EW} \mid \mathsf{MMS}) \geq \frac{\frac{1}{2}}{\frac{1}{n} + 2\epsilon} \to \frac{n}{2}, \text{ when } \epsilon \to 0,$$

which finishes the proof. □

Remark: If we have an *oracle* on computing the egalitarian welfare-maximizing allocation, then the MMS allocation achieving $\frac{n}{2}$ PoF ratio can be computed in polynomial time.

3.2 On utilitarian welfare

For utilitarian welfare, we also utilize ALG-M(β) and show an asymptotically tight PoF ratio of $\Theta(n)$. In particular, by setting $\beta = -\frac{2}{n}$, one can compute, in polynomial time, an MMS allocation achieving utilitarian welfare at least -3.

Theorem 12. *For utilitarian welfare and MMS fairness, the price of fairness is* $\Theta(n)$.

Proof. For the upper bound, if $\mathsf{OPT}_U \geq -\frac{1}{n}$, then the utilitarian-welfare maximizing allocation is also MMS. Thus it suffices to consider the case where $\mathsf{OPT}_U < -\frac{1}{n}$.

Let $\mathbf{A} = (A_1, \ldots, A_n)$ be the allocation returned by ALG-M($-\frac{2}{n}$). Without loss of generality, agents are renumbered by the order of receiving bundles; that is, agent 1 is the first to receive a bundle, while agent n is the last. By Lemma 7, allocation $\mathbf{A}$ is MMS. Let N_1 and N_2 be the sets of agents whose bundles are assigned in Step 6 or 9, respectively. Then, by Step 5, $v_i(A_i) \geq -\frac{2}{n}$ holds for all $i \in N_1$. As for N_2, denote by $N_2 = \{i_1, i_2, \ldots, i_p\}$, and bundle A_{i_l} is on the left of A_{i_k} for any $l < k \leq p$. By the selection of agents in Step 8, $v_{i_p}(A_{i_k}) \leq v_{i_k}(A_{i_k})$ for $k \leq p$. Thus, the total value of agents in N_2 is

$$\sum_{i \in N_2} v_i(A_i) = \sum_{k \in [p]} v_{i_k}(A_{i_k}) \geq \sum_{k \in [p]} v_{i_p}(A_{i_k}) \geq -1,$$

where the last inequality is due to normalized valuations. Accordingly, the welfare of $\mathbf{A}$ is bounded by

$$\begin{aligned}\sum_{i \in [n]} v_i(A_i) &\geq \sum_{i \in N_1} v_i(A_i) + \sum_{i \in N_2} v_i(A_i) + v_n(A_n) \\ &\geq -\frac{2}{n} \cdot |N_1| - 1 - \frac{2}{n} \\ &\geq -3,\end{aligned}$$

where the second inequality comes from Corollary 10 and the last inequality is due to $|N_1| \leq n - 1$. Thus, in polynomial time, we find an MMS allocation with welfare at least -3, and, moreover, the price of MMS is at most $3n$.

As for the lower bound, consider an instance with n agents and a set $E = \{e_1, \ldots, e_{3n-2}\}$ of $3n - 2$ chores. The valuations are shown in Table 3. One can verify that $\mathsf{MMS}_i = -\frac{1}{n}$ for all $i \in N$. In a

Chores	e_1	$\cdots$	e_{2n}	e_{2n+1}	$\cdots$	e_{3n-2}
$v_1(\cdot)$	$-\frac{1}{n^2}$	$\cdots$	$-\frac{1}{n^2}$	$-\frac{1}{n}$	$\cdots$	$-\frac{1}{n}$
$v_i(\cdot)$, $i \geq 2$	$-\frac{1}{2n}$	$\cdots$	$-\frac{1}{2n}$	0	$\cdots$	0

Table 3: The lower bound instance in Theorem 12

utilitarian welfare-maximizing allocation $\mathbf{O} = (O_1, \ldots, O_n)$, the first $2n$ items are assigned to agent 1, and remaining items are, in a contiguous way, arbitrarily allocated to other agents so that $v_1(O_1) = -\frac{2}{n}$, $v_i(O_i) = 0$ for $i \geq 2$ and $\mathsf{UW}(\mathbf{O}) = -\frac{2}{n}$. However, this allocation is not MMS fair for agent 1 since $v_1(O_1) < \mathsf{MMS}_1$. For any MMS allocation $\mathbf{A}$, agent 1 can receive at most n of the first $2n$ items, and thus, at least n of the first $2n$ items are assigned to agents $i \geq 2$. Thus, $\mathsf{UW}(\mathbf{A}) \leq -\frac{1}{n} - \frac{1}{2}$, and consequently, the price of fairness is at least

$$\mathsf{PoF}(\mathsf{UW} \mid \mathsf{MMS}) \geq \frac{\frac{1}{n} + \frac{1}{2}}{\frac{2}{n}} = \frac{1}{2} + \frac{n}{4} = \Omega(n).$$

Therefore, the price of MMS with respect to utilitarian welfare is $\Theta(n)$. □

4 PRICES OF PROP1 FOR GENERAL $n \geq 3$

In this section, we quantify efficiency loss under PROP1 allocations. Similar to Section 3, we introduce another parametric algorithm ALG-P(β) (see Algorithm 2) which leverages the protocol of moving-knife while carefully handling the item right-connected to the underlying bundle (from the left end to the knife position). Then algorithm ALG-P(β) can compute PROP1 allocations, ensuring that every agent's value remains above a certain real number. In the following, we first demonstrate that ALG-P(β) can return PROP1 allocations for all $\beta \leq -\frac{2}{n}$ and then establish tight PoF ratios by implementing Algorithm 2 with properly chosen β. Although the approach here resembles that of MMS, the analysis needs to be more detailed.

Algorithm 2 ALG-P(β)

Input: An instance $I = \langle N, E, \{v_i\}_{i=1}^n \rangle$ and a real number β.
Output: An allocation $\mathbf{A} = (A_1, \ldots, A_n)$.

1: Initialize $N_0 \leftarrow N$, $E_0 \leftarrow E$ and $A_i = \emptyset$ for all $i \in N$.
2: **while** $|N_0| > 1$ & $E_0 \neq \emptyset$ **do**
3: Let $e_L \in E$ be the left most item.
4: **if** there exists an agent $i \in N_0$ such that $v_i(e_L) \geq -\frac{1}{n}$ **then**
5: Suppose p be the largest index such that there exists some agent i with $v_i(L(p) \cap E_0) \geq -\frac{1}{n}$. Let $\widehat{N}_0 = \{i \in N_0 \mid v_i(L(p) \cap E_0) \geq -\frac{1}{n}\}$. If $L(p) \cap E_0 = E_0$, then assign E_0 to an arbitrary agent in $\widehat{N}_0$. If $L(p) \cap E_0 \subsetneq E_0$, then let $i^* \in \arg\max_{i \in \widehat{N}_0} v_i(L(p+1) \cap E_0)$;
6: **if** $v_{i^*}(L(p+1) \cap E_0) < \beta$ **then**
7: $A_{i^*} \leftarrow L(p) \cap E_0$;
8: **else**
9: $A_{i^*} \leftarrow L(p+1) \cap E_0$;
10: **end if**
11: **else**
12: Let $i^* \in \arg\max_{i \in N_0} v_i(e_L)$, where ties are broken arbitrarily and assign $A_{i^*} \leftarrow \{e_L\}$;
13: **end if**
14: Update $N_0 \leftarrow N_0 \setminus \{i^*\}$ and $E_0 \leftarrow E_0 \setminus A_{i^*}$;
15: **end while**
16: If $E_0 \neq \emptyset$, assign E_0 to an arbitrary agents in N_0.
17: **return A**

Lemma 13. *For any $\beta \leq -\frac{2}{n}$,* ALG-P(β) *returns a* PROP1 *allocation in polynomial time.*

Proof. In every round of the while-loop of Algorithm 2, the number of agents is reduced by one, and hence the algorithm terminates in $O(mn)$ time. Without loss of generality, agents are ordered by the order of receiving bundles in the algorithm; that is, agent 1 is the first to receive a bundle and agent n is the last.

Denote by $\mathbf{A} = (A_1, \ldots, A_n)$ the allocation returned by ALG-P(β). Given an agent i with $i < n$, if agent i receives A_i in the while-loop, then it is not hard to verify that she satisfies PROP1. Accordingly, if the while-loop stops as all chores are assigned, the statement is proved. Accordingly, the remaining work is to show when the while-loop terminates as $E_0 \neq \emptyset$ (and hence $|N_0| = 1$), agent n (the last agent) can still be PROP1. It suffices to show that, after assigning bundles $A_1, A_2, \ldots, A_{n-1}$, agent n's value on remaining items is at least $-\frac{1}{n}$.

Fix i with $i \leq n-1$. If A_i is assigned in Steps 9 or 12, then $v_n(A_i) < -\frac{1}{n}$; otherwise, violating Steps 5 or 4. Note that for those A_i's allocated in Step 7, both $v_n(A_i) < -\frac{1}{n}$ and $v_n(A_i) \geq -\frac{1}{n}$ are possible. Denote by $\mathcal{P}_n = \{i \in [n] \mid A_i$ is assigned in Step 7 and $v_n(A_i) \geq -\frac{1}{n}\}$. Then, by the construction, for any $j < n$, if $j \notin \mathcal{P}_n$, then $v_n(A_j) < -\frac{1}{n}$ holds. For any $j \in \mathcal{P}_n$, let e^j be the item on the right of A_j and $\{e^j\} \cup A_j \in \mathcal{C}$. Then, according to Steps 5 and 6, it holds that $v_n(A_j \cup \{e^j\}) \leq v_j(A_j \cup \{e^j\}) < \beta \leq -\frac{2}{n}$. We next let $\mathcal{Q}_n$ be the set of agents whose bundles are on the right of and connected to some A_j with $j \in \mathcal{P}_n$. Formally, $\mathcal{Q}_n = \{t \in [n] \mid A_t$ is on the right of A_j and $A_t \cup A_j \in \mathcal{C}$ for some $j \in \mathcal{P}_n\}$.

Claim 14. *$\mathcal{P}_n \cap \mathcal{Q}_n = \emptyset$ and $n-1 \notin \mathcal{P}_n$.*

By Claim 14, there is one-to-one correspondence between $\mathcal{P}_n$ and $\mathcal{Q}_n$, and moreover, $\mathcal{P}_n \cup \mathcal{Q}_n \subseteq [n-1]$. Then, we provide an upper bound of agent n's value on allocated items;

$$\begin{aligned}\sum_{i=1}^{n-1} v_n(A_i) &= \sum_{i \in \mathcal{P}_n \cup \mathcal{Q}_n} v_n(A_i) + \sum_{i \in [n-1] \setminus \mathcal{P}_n \cup \mathcal{Q}_n} v_n(A_i) \\ &< -\frac{|\mathcal{P}_n \cup \mathcal{Q}_n|}{2} \cdot \beta - \frac{n-1-|\mathcal{P}_n \cup \mathcal{Q}_n|}{n} \\ &\leq -\frac{n-1}{n},\end{aligned}$$

where the last inequality transition is due to $\beta \leq -\frac{2}{n}$. As $v_n(E) = -1$, we have $v_n(A_n) > -\frac{1}{n}$, and therefore, allocation $\mathbf{A}$ is PROP1. ☐

The proof of Lemma 13 also implies that agent n (the last agent) receives a value at least $-\frac{1}{n}$.

Corollary 15. *For any $\beta \leq -\frac{2}{n}$, agent n (the last agent to receive a bundle), in the allocation returned by* ALG-P(β)*, has a value at least $-\frac{1}{n}$.*

4.1 On egalitarian welfare

We now present the tight PoF ratio regarding egalitarian welfare. We examine the allocation returned by ALG-P($-\frac{1}{2}$). Similar to MMS, we prove that either the allocation returned by ALG-P($-\frac{1}{2}$) is a PROP1 allocation achieving the target PoF ratio $\frac{n}{2}$, or it can serve as a starting point for obtaining the desired PROP1 allocation through bundle reallocation. However, such a reallocation is not always straightforward and may pose challenges. Different reallocation approaches are needed to address various cases.

Before presenting our main result, we establish a sufficient condition for obtaining a partial PROP1 allocation.

Lemma 16. *Suppose $S \subseteq E$ be a connected bundle and $\widetilde{N} \subseteq N$ be a set of agents. If for any $i \in \widetilde{N}$, $v_i(S) \geq -\frac{|\widetilde{N}|}{|N|}$, then S can be assigned to $\widetilde{N}$, satisfying* PROP1 *requirement defined on an instance with agents set being N.*

Proof Sketch. One can think of assigning S to $\widetilde{N}$ in the following way: repeatedly identify the farthest chore for which at least one agent values the chore from the left end to the current position at no less than $-\frac{1}{|N|}$, and then allocate the selected bundle, along with the next item (if any), to the chosen agent. The formal proof is presented in the full version. ☐

In essence, the lemma above suggests that when dealing with a reduced instance I' involving a smaller number of agents (e.g., six) and a connected bundle S, if each of these agent values S at least $-\frac{6}{n}$, then it becomes feasible to allocate S to these six agents, meeting the PROP1 criteria of the original instance I, which has n agents. We remark that to establish the exact PoF ratio, we have a unified approach to find the desired PROP1 allocation for all $n \geq 8$. Unfortunately, such an approach does not carry over to the case of $3 \leq n \leq 7$. And finding the desired PROP1 allocation for $n \leq 7$ involves detailed analysis of cases.

Theorem 17. *For egalitarian welfare and* PROP1*, the price of fairness is 2 when $n = 3$ and is $\frac{n}{2}$ when $n \geq 4$.*

Proof. We here prove the upper bound for the case of $n \geq 8$. The upper bound proof for $3 \leq n \leq 7$ is deferred to the full version. If $\mathsf{OPT}_E \geq -\frac{1}{n}$, then an egalitarian welfare maximizing allocation is PROP1 and the statement is proved. Thus, we can further assume $\mathsf{OPT}_E < -\frac{1}{n}$. Also, if $\mathsf{OPT}_E \leq -\frac{2}{n}$, then as $v_i(E) = -1$ for all $i \in [n]$, any PROP1 allocation results in the PoF ratio of $\frac{n}{2}$. Thus, we can focus on the case where $-\frac{1}{2} \leq -\frac{2}{n} < \mathsf{OPT}_E < -\frac{1}{n}$.

Denote by $\mathbf{A} = (A_1, \ldots, A_n)$ the allocation returned by ALG-P$(-\frac{1}{2})$. Without loss of generality, agents are renumbered by the order of receiving bundles in the algorithm; that is, agent 1 is the first to receive a bundle and agent n is the last. If $\mathsf{EW}(\mathbf{A}) \geq -\frac{1}{2}$, then the statement is proved. If $\mathsf{EW}(\mathbf{A}) < -\frac{1}{2}$, let agent k be the one such that $v_k(A_k) \leq v_i(A_i)$ for $i \in [n]$, which implies $v_k(A_k) = \mathsf{EW}(\mathbf{A}) < -\frac{1}{2}$. By Steps 5, 7 and 9, one can verify that A_k is assigned in Step 12 and moreover $v_j(A_k) \leq v_k(A_k) < -\frac{1}{2}$ for all $j > k$. Let k' be the index such that $A_{k'}$ is on the left of A_k and $A_{k'} \cup A_k \in C$. Note that $A_{k'}$ is guaranteed to exist; otherwise, contradicting $\mathsf{OPT}_E > -\frac{1}{2}$. Denote by $\mathcal{R} = \{i \in [n] \mid A_i \neq \emptyset\}$ and $\mathcal{J} = [n] \setminus \mathcal{R}$. Note that bundle $A_{|\mathcal{R}|}$ is the right-most non-empty one among $\{A_j\}_{j=1}^n$.

We now analyze agent n's value on bundles. For any $j \in \mathcal{R}$ but $j \neq |\mathcal{R}|, k', k$, we claim that $v_n(A_j) < -\frac{1}{n}$; if not, A_j must be assigned in Step 7, and then $v_n(A_j \cup A_{j+1}) < \beta = -\frac{1}{2}$. As $j \neq k'$ (hence $j + 1 \neq k$), and thus, $v_n(E) \leq v_n(A_j \cup A_{j+1}) + v_n(A_k) < -1$, a contradiction. Then, we have the following

$$v_n(E) = v_n(A_k \cup A_{k'} \cup A_{|\mathcal{R}|}) + \sum_{j \in \mathcal{R},\ j \neq k', k, |\mathcal{R}|} v_n(A_j) < -\frac{1}{2} - \frac{|\mathcal{R}| - 3}{n},$$

where the last inequality transition is due to $v_n(A_j) < -\frac{1}{n}$ for all $j \in \mathcal{R}$ and $j \neq k', k, |\mathcal{R}|$. Due to normalized valuations, the above inequality implies $|\mathcal{R}| < \frac{n}{2} + 3$, and hence $n - |\mathcal{R}| > \frac{n}{2} - 3$, meaning that the number of agents receiving empty bundles is larger than $\frac{n}{2} - 3$, i.e., $|\mathcal{J}| > \frac{n}{2} - 3$.

As $\mathsf{OPT}_E \geq -\frac{1}{2}$, there exists an agent p^* such that $v_{p^*}(A_k) \geq -\frac{1}{2}$, and moreover, by ALG-P$(-\frac{1}{2})$, we know $q^* < k$. Consider a partial allocation $\mathbf{B}$, in which A_k is assigned to agent p^* and $B_j = A_j$ for all $j \in \mathcal{R}$ and $j \neq p^*, k$. Note that agent p^* is PROP1 as $|A_k| = 1$ and agent $j \in \mathcal{R} \setminus (\{k\} \cup \{p^*\})$ is PROP1 since $\mathbf{A}$ is PROP1. The unallocated items are A_{p^*} and moreover for each $j \in \mathcal{J}$, agent j's value on A_{p^*} is larger than $-\frac{1}{2}$ so that assigning A_{p^*} to agents in $\mathcal{J}$ does not violate the requirement of $\mathsf{EW} > -\frac{1}{2}$.

We next show that A_{p^*} can be assigned to $\mathcal{J}$ without violating PROP1. Note that each agent $j \in \mathcal{J}$ is equivalent to agent n as every agent in $\mathcal{J}$ gets an empty bundle in $\mathbf{A}$. Consequently, for any $j \in \mathcal{J}$, her value on A_{p^*} can be bounded as follows;

$$v_j(A_{p^*}) = v_l(E) - v_j(A_{k'} \cup A_k \cup A_{|\mathcal{R}|}) - \sum_{\substack{t \in \mathcal{R} \\ t \neq p^*, k', k, |\mathcal{R}|}} v_j(A_t) > -\frac{1}{2} + \frac{|\mathcal{R}| - 4}{n},$$

where the inequality transition is due to $v_j(A_t) < -\frac{1}{n}$ for all $t \in \mathcal{R}$ and $t \neq k', k, |\mathcal{R}|$. As $|\mathcal{J}| = n - |\mathcal{R}|$, we have $-\frac{|\mathcal{J}|}{n} \leq -\frac{1}{2} + \frac{|\mathcal{R}|-4}{n}$ for $n \geq 8$. According to Lemma 16, assigning A_{p^*} to agents in $\mathcal{J}$ extends $\mathbf{B}$ to a complete PROP1 allocation.

For the lower bound regarding $n \geq 4$, consider an instance with n agents and a set $E = \{e_1, \ldots, e_{n+2}\}$ of n+2 chores. The valuations are shown in the following table, where $\epsilon > 0$ is arbitrarily small. In an egalitarian welfare maximizing allocation $\mathbf{O}$, agent 1 receives bundle $O_1 = \{e_1, e_2, e_3\}$ and has a value $-\frac{1}{n} - 2\epsilon$, and agents $i \geq 2$ receives other items and have value zero. Then, we have $\mathsf{OPT}_E = -\frac{1}{n} - 2\epsilon$. However, in allocation $\mathbf{O}$, agent 1 is not satisfied regarding PROP1 as removing item e_1 or e_3 still yields value $-\frac{1}{n} - \epsilon < -\frac{1}{n}$ for him, and thus, in any PROP1 allocation, agent 1 cannot receive all e_1, e_2, e_3. Hence, any PROP1 allocation $\mathbf{A}$ must allocate at least one of e_1, e_3 to agents $i \geq 2$, which implies $\mathsf{EW}(\mathbf{A}) \leq -\frac{1}{2}$. Therefore, the price of PROP1 with respect to egalitarian welfare is at least

Items	e_1	e_2	e_3	e_4	$\cdots$	e_n	e_{n+1}	e_{n+2}
$v_1(\cdot)$	$-\epsilon$	$-\frac{1}{n}$	$-\epsilon$	$-\frac{1}{n}$	$\cdots$	$-\frac{1}{n}$	$-\frac{1}{n}$	$-\frac{1}{n} + 2\epsilon$
$v_i(\cdot)$ for $i \geq 2$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\cdots$	0	0	0

Table 4: The Lower Bound Instance for $n \geq 4$

$$\mathsf{PoF}(\text{PROP1} \mid \mathsf{EW}) \geq \frac{-\frac{1}{2}}{-\frac{1}{n} - 2\epsilon} \to \frac{n}{2}, \text{ when } \epsilon \to 0,$$

which completes the proof. □

Note that Sun and Li [40] show that in the model of allocating contiguous blocks of indivisible goods, PROP1 allocations, in the worst-case scenario, do not guarantee any egalitarian welfare, i.e., the price of PROP1 is infinite for goods. On the other hand, Theorem 17 of this work indicates that PROP1 allocations, in the context of contiguous chores, can guarantee a certain degree of egalitarian welfare. This observation underscores the notable contrast between goods and chores, even within the framework of allocating contiguous blocks of items.

Remark: If we have an *oracle* on computing the egalitarian welfare-maximizing allocation, then the PROP1 allocation achieving $\frac{n}{2}$ PoF ratio can be computed in polynomial time.

4.2 On utilitarian welfare

For utilitarian, we also provide a tight ratio on the price of PROP1. The proof relies on the allocation returned by Algorithm 2 with $\beta = -\frac{2}{n}$.

THEOREM 18. *For utilitarian welfare and* PROP1*, the price of fairness is* $\Theta(n)$.

PROOF. We begin with the upper bound part. Denote by $\mathbf{A} = (A_1, \ldots, A_n)$ the allocation returned by ALG-P($-\frac{2}{n}$), and according to Lemma 13, allocation $\mathbf{A}$ is PROP1. Denote by N_1, N_2, N_3 be, respectively, the set of agents who receive items in Steps 7, 9, and 12. Then $v_i(A_i) \geq -\frac{1}{n}$ holds for all $i \in N_1$ and $v_i(A_i) \geq -\frac{2}{n}$ holds for all $i \in N_2$. We now bound the value of agents in N_3. Suppose $N_3 = \{i_1, i_2, \ldots, i_p\}$ and without loss of generality bundle A_{i_l} is on the left of bundle A_{i_k} for any $l < k \leq p$. For every $k \leq p$, we have $v_{i_p}(A_k) \leq v_{i_k}(A_{i_k})$ due to the condition in Step 12 of Algorithm 2. Consequently, the welfare of agents in N_3 is bounded by

$$\sum_{i \in N_3} v_i(A_i) \geq \sum_{i \in N_3} v_{i_p}(A_i) \geq -1,$$

where the last transition is due to normalized valuations. As for agent n, according to the proof of Lemma 13, we have $v_n(A_n) \geq -\frac{1}{n}$. Therefore, the utilitarian welfare of allocation $\mathbf{A}$ satisfies the following,

$$\begin{aligned}\mathsf{UW}(\mathbf{A}) &\geq (\sum_{i \in N_1} + \sum_{i \in N_2} + \sum_{i \in N_3}) v_i(A_i) + v_n(A_n) \\ &> -\frac{2}{n}(|N_1| + |N_2|) - 1 - \frac{1}{n} \\ &\geq -3,\end{aligned}$$

where the second inequality transition is due to Corollary 15 and the last inequality transition is due to the fact that $|N_1|+|N_2| \leq n-1$ holds. Suppose $\mathbf{O}$ be a contiguous utilitarian welfare-maximizing allocation. If $\mathsf{UW}(\mathbf{O}) \geq -\frac{1}{n}$, then allocation $\mathbf{O}$ is PROP1 and the statement trivially holds. We can further assume $\mathsf{UW}(\mathbf{O}) < -\frac{1}{n}$. As allocation $\mathbf{A}$ is a PROP1 allocation with welfare at least -3, the price of PROP1 is at most $3n$.

As for the lower bound, consider an instance with n (even) agents and a set $E = \{e_1, \ldots, e_{n+1}\}$ of $n+1$ chores. The valuations are shown in Table 5. In a utilitarian welfare maximizing allocation $\mathbf{O} = (O_1, \ldots, O_n)$, the first n chores are assigned to agent 1 and chore e_{n+1} is arbitrarily allocated to the other agents so that $\mathsf{UW}(\mathbf{O}) = -\frac{3n+2}{n(n+1)}$. However, agent 1 violates PROP1 in $\mathbf{O}$. In a PROP1 allocation $\mathbf{A}$, agent 1 can receive at most $\frac{n}{2}+1$ items, which leaves at least n items to other agents. Thus, $\mathsf{UW}(\mathbf{A}) \leq -\frac{n}{2(n+1)} - \frac{n+2}{n^2}$, and consequently, the price of PROP1 has the following lower bound

Items	e_1	e_2	$\cdots$	e_n	e_{n+1}
$v_1(\cdot)$	$-\frac{2}{n^2}$	$-\frac{2}{n^2}$	$\cdots$	$-\frac{2}{n^2}$	$-\frac{n-2}{n}$
$v_i(\cdot)$ for $i \geq 2$	$-\frac{1}{n+1}$	$-\frac{1}{n+1}$	$\cdots$	$-\frac{1}{n+1}$	$-\frac{1}{n+1}$

Table 5: The Lower Bound Instance

$$\mathsf{PoF}(\mathsf{PROP1} \mid \mathsf{UW}) \geq \frac{n^3 + 2(n+2)(n+1)}{2n(3n+2)} \geq \frac{n}{6} + \frac{2}{9} = \Omega(n),$$

which completes the proof. □

5 TWO AGENTS

The allocation problem involving two agents is also of considerable significance and has garnered notable attention [1, 2]. In this section, we delve into the PoF ratios when $n = 2$. Unlike the general case, it's noteworthy that both MMS and PROP1 fairness concepts are compatible with optimal egalitarian welfare when there are only two agents involved.

THEOREM 19. *When* $n = 2$*, for both* MMS *and* PROP1*, the price of fairness with respect to egalitarian welfare is 1; the price of fairness with respect to utilitarian welfare is* 2.

In order to prove Theorem 19, we need the following lemma.

LEMMA 20. *There exists an allocation that satisfies MMS, PROP1, and attains optimal egalitarian welfare and achieves utilitarian welfare at least -1.*

PROOF SKETCH. We consider an allocation $\mathbf{O}$ constructed as follows; $\mathbf{O}$ first maximizes the egalitarian welfare among all allocations; If there is a tie, $\mathbf{O}$ minimizes the number of items allocated to the agent with a smaller value. By detailed analysis of agents' value, one can show that allocation $\mathbf{O}$ is both MMS and PROP1 and achieving utilitarian welfare at least -1. The formal proof is deferred to the full version. □

Now we are ready to prove Theorem 19.

PROOF OF THEOREM 19. Consider the allocation $\mathbf{O}$ constructed in Lemma 20. The PoF regarding egalitarian welfare is straightforward by the design. We focus on utilitarian welfare in the following. Note that $\mathsf{OPT}_E \geq \mathsf{OPT}_U$ always holds. Since $\mathbf{O}$ achieves the optimal egalitarian welfare, we have $\mathsf{UW}(\mathbf{O}) \geq 2\mathsf{OPT}_E \geq 2\mathsf{OPT}_U$ where the first inequality transition is due to $n = 2$. Thus, the price of fairness ratio regarding utilitarian welfare is at least 2. The lower bound instances that match the ratio are provided in the full version. □

6 CONCLUDING REMARK

In this work, we revisited fairness and efficiency trade-off in the model of allocating contiguous blocks of indivisible chores. We focused on the fairness notions of MMS and PROP1, of which the existence is guaranteed in the underlying model. We utilize the well-studied notion of price of fairness to quantify the social welfare loss under fairness allocations. For every pairwise fairness and welfare combination, we establish the tight ratio of the price of fairness.

We also discussed, in the full version, the price of fairness with respect to other fairness criteria such as envy-free (or equitability) up to one item [7, 19, 26, 39] whose existence however in the connectivity constraint setting is still unknown. We found out that these two fair allocations can not provide a bounded welfare guarantee in some hard instances. We hope that our results on the price of fairness may shed some light on fairness and efficiency trade-off and be helpful in guiding the decision-maker to pick the proper underlying fairness notions.

ACKNOWLEDGMENTS

This work is funded by GDSTC (No. 2023A1515010592), NSFC (No. 62102333) and HKSAR RGC (No. PolyU 25211321).

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