NP$^{\text{PP}}$-Completeness of Control by Adding Players to Change the Penrose–Banzhaf Power Index in Weighted Voting Games

Extended Abstract

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ABSTRACT

Weighted voting games are an important class of simple games that can be compactly represented and have many real-world applications. Rey and Rothe [14] introduced the notion of structural control by adding players to or deleting them from weighted voting games, with the goal to either change or maintain a given player's power in a given game with respect to the (probabilistic) Penrose–Banzhaf power index [4] or the Shapley–Shubik power index [17]. For control by adding players, they showed PP-hardness as the best known lower bound and an upper bound of NP$^{\text{PP}}$, where PP is "probabilistic polynomial time." We optimally improve their results by establishing NP$^{\text{PP}}$-hardness (and thus NP$^{\text{PP}}$-completeness) of all problems related to the Penrose–Banzhaf index and for the problem of maintaining the Shapley–Shubik index when players are added.

KEYWORDS

weighted voting games; computational complexity; game theory

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1 INTRODUCTION

Weighted voting games, an important class of simple, compactly representable coalitional games, have many real-world applications. For instance, one can use them to model and analyze collective decision-making in legislative bodies and in parliamentary voting [16], such as the European Union or the International Monetary Fund [5], in joint stock companies, etc.

A very important subject of the research on weighted voting games is the analysis of how significant players are in these games, i.e., what their impact is in forming winning coalitions. To measure this impact, power indices have been proposed, including the *normalized Penrose–Banzhaf index* due to Penrose [12] and Banzhaf [2], the *probabilistic Penrose–Banzhaf index* due to Dubey and Shubik [4], and the *Shapley–Shubik index* due to Shapley and Shubik [17]. We are concerned with the latter two power indices.

Various ways of how to rig a given player's power in a weighted voting game have been studied. For example, Aziz et al. [1] and Rey and Rothe [13] investigated the impact of merging or splitting players, the latter being also known as "false-name manipulation." Zuckerman et al. [18] studied how to influence the power of players by manipulating the quota in weighted voting games. Later on, Rey and Rothe [14] introduced control by adding players to or by deleting them from a weighted voting game. Recently, their complexity results have been improved by Kaczmarek and Rothe [8]. Kaczmarek et al. [9] studied control by adding or deleting edges in graph-restricted weighted voting games.

Continuing the work of Rey and Rothe [14] on the complexity of control by adding players to a weighted voting game, we optimally improve their PP-hardness for these problems to NP$^{\text{PP}}$-hardness, thus obtaining a lower bound that matches their upper bound of NP$^{\text{PP}}$. This establishes NP$^{\text{PP}}$-completeness for the problems with the goal of increasing, nondecreasing, decreasing, nonincreasing, or maintaining the probabilistic Penrose–Banzhaf index and for the problem with the goal of maintaining the Shapley–Shubik index.

Note that the problems related to merging players in weighted voting games were shown to be PP-complete [13]. However, the complexity gap between PP-hardness and membership in NP$^{\text{PP}}$ also persists for false-name manipulation [13] as well as for some of the problems related to control by adding or deleting edges in graph-restricted weighted voting games [9]. Thus the techniques we developed may turn out useful for these open problems as well.

2 PRELIMINARIES

A *coalitional game* is a pair (N, v), where $N = \{1, \dots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}_{\geq 0}$ is the characteristic function. A *coalition* is a subset of N. (N, v) is said to be *simple* if v is *monotonic* (i.e., $v(C) \leq v(D)$ for all C, D with $C \subseteq D \subseteq N$), and $v(C) \in \{0, 1\}$ for each $C \subseteq N$. A *weighted voting game (WVG)* $\mathcal{G} = (w_1, \dots, w_n; q)$ is a simple coalitional game (N, v) that is represented by the players' weights, with w_i being the weight of player $i \in N$, and a nonnegative integer quota q. For each coalition $C \subseteq N$, we say C is a *winning coalition* if $v(C) = 1$ (i.e., $w_C \geq q$), and C is a *losing coalition* if $v(C) = 0$ (i.e., $w_C < q$), where $w_C = \sum_{i \in C} w_i$. We call a player $i \in N$ *pivotal for coalition* $C \subseteq N \setminus \{i\}$ if $v(C \cup \{i\}) - v(C) = 1$.

The influence or significance of players in a given game is usually measured by so-called *power indices.* Two of the most popular and well-known power indices are the *probabilistic Penrose–Banzhaf*

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power index (introduced by Dubey and Shapley [4] as an alternative to the original *normalized Penrose–Banzhaf index* [2, 12]) and the *Shapley–Shubik power index* (introduced by Shapley and Shubik [17]). For any player i in $\mathcal{G}$, the former is defined by

$$\beta(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))}{2^{n-1}},$$

and the latter is defined by

$$\varphi(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} |S|!(n-1-|S|)!(v(S \cup \{i\}) - v(S))}{n!}.$$

We assume the reader to be familiar with the fundamental concepts of computational complexity theory (see, e.g., [6, 11, 15]), such as the well-known complexity classes P (*deterministic polynomial time*), NP (*nondeterministic polynomial time*), and PP (*probabilistic polynomial time* [7]), and with the notions of completeness and hardness for a complexity class with respect to the polynomial-time many-one reducibility. $\mathrm{NP}^{\mathrm{PP}}$ is the class of problems solvable by an NP oracle Turing machine that has access to a PP oracle.

While Gill [7] proved that MajSAT = $\{\phi \mid \phi$ is a boolean formula satisfied by a majority of truth assignments$\}$ is a standard PP-complete problem, Littman et al. [10] introduced the following problem and proved that it is $\mathrm{NP}^{\mathrm{PP}}$-complete:

E-Majority-SAT (E-MajSAT)	
Given:	A boolean formula ϕ with n variables $x_1, \ldots, x_n$ and an integer k, $1 \leq k \leq n$.
Question:	Does there exist an assignment to the first k variables $x_1, \ldots, x_k$ such that a majority of assignments to the remaining $n - k$ variables $x_{k+1}, \ldots, x_n$ satisfies ϕ?

If we change “a majority” into “at most half” in the question above, we obtain another $\mathrm{NP}^{\mathrm{PP}}$-complete problem, E-Minority-SAT (E-MinSAT), which was introduced by de Campos et al. [3].

Rey and Rothe [14] defined problems capturing control by adding players to a given WVG so as to change a given player's power in the modified game. To increase this power for an index PI, the control problem is defined as follows:

Control-by-Adding-Players-to-Increase-PI	
Given:	A WVG $\mathcal{G}$ with a set N of players, a set M of players (given by their weights) that can be added to $\mathcal{G}$, a distinguished player $p \in N$, and a positive integer $k \leq \|M\|$.
Question:	Can at most k players $M' \subseteq M$ be added to $\mathcal{G}$ such that for the new game $\mathcal{G}_{\cup M'}$, it holds that $\mathrm{PI}(\mathcal{G}_{\cup M'}, p) > \mathrm{PI}(\mathcal{G}, p)$?

Changing “>” in the question above to “<,” “≤,” “≥,” and “=,” respectively, we analogously obtain the corresponding control problems for decreasing, nonincreasing, nondecreasing, and maintaining PI. In case of nondecreasing, nonincreasing, and maintaining PI we additionally assume that at least one new player is added.

Rey and Rothe [14] showed PP-hardness of these five control problems for both the Penrose–Banzhaf and the Shapley–Shubik power index, and they identified $\mathrm{NP}^{\mathrm{PP}}$ as their best known upper bound. We aim at raising their PP-hardness lower bound to $\mathrm{NP}^{\mathrm{PP}}$-hardness, thus establishing their completeness in this class.

Finally, we introduce yet another $\mathrm{NP}^{\mathrm{PP}}$-complete problem, which we use in our proofs:

E-Exact-SAT	
Given:	A boolean formula ϕ with n variables $x_1, \ldots, x_n$, an integer k, $1 \leq k \leq n$, and an integer ℓ.
Question:	Is there an assignment to the first k variables $x_1, \ldots, x_k$ such that *exactly* ℓ assignments to the remaining $n - k$ variables $x_{k+1}, \ldots, x_n$ satisfy ϕ?

Lemma 1. *E-Exact-SAT is* $\mathrm{NP}^{\mathrm{PP}}$*-complete.*

3 $\mathrm{NP}^{\mathrm{PP}}$-HARDNESS OF CONTROL BY ADDING PLAYERS TO A WEIGHTED VOTING GAME

To show $\mathrm{NP}^{\mathrm{PP}}$-hardness of our control problems, we use some functions that convert an input of E-MajSAT, E-MinSAT, or E-Exact-SAT into weight vectors for some of the players defined for an input of one of our control problems. We present an example of such a conversion in the definition and its properties used in our proofs as follows: Let ϕ be a boolean formula in CNF with variables $x_1, \ldots, x_n$ and m clauses. Let k be an integer with $1 \leq k \leq n$ and set $r = \lceil \log_2 n \rceil - 1$. For some integer t satisfying $10^t > 2^{\lceil \log_2 n \rceil + 1}$, we define the function F mapping ϕ to the $(2n + m(r+1))$-tuple $(a_1, \ldots, a_n, b_1, \ldots, b_n, c_{1,0}, \ldots, c_{1,r}, \ldots c_{m,0}, \ldots, c_{m,r})$, where for each variable x_i, $i \in \{1, \ldots, n\}$,

$$\begin{aligned} a_i &= 10^{t(m+1)+i} + \sum_{\substack{j\,:\,\text{clause } j \\ \text{contains } x_i}} 10^{tj}, \\ b_i &= 10^{t(m+1)+i} + \sum_{\substack{j\,:\,\text{clause } j \\ \text{contains } \neg x_i}} 10^{tj} \end{aligned}$$

and each $j \in \{1, \ldots, m\}$ and each $s \in \{0, \ldots, r\}$, $c_{j,s} = 2^s \cdot 10^{tj}$. Moreover, let $q_F = \left(\sum_{i=1}^{n} 10^{t(m+1)+i}\right) + 2^{\lceil \log_2 n \rceil} \sum_{j=1}^{m} 10^{tj}$. The conversion has the following property.

Lemma 2. *Let ϕ be a boolean formula in CNF with n variables. There exists a bijection from the set of truth assignments to the variables $x_1, \ldots, x_n$ satisfying ϕ into the set of sublists of $F(\phi)$ whose total value is q_F.*

Theorem 1. *The problems of control by adding players to a given WVG so as to increase, nondecrease, decrease, nonincrease, or maintain a given player's Penrose–Banzhaf index or to maintain her Shapley–Shubik index are* $\mathrm{NP}^{\mathrm{PP}}$*-hard.*

4 CONCLUSIONS

We have shown that control by adding players to WVGs so as to increase, nondecrease, decrease, nonincrease, or maintain a given player's Penrose–Banzhaf power index or to maintain a given player's Shapley–Shubik power index is complete for $\mathrm{NP}^{\mathrm{PP}}$, thus settling the complexity of these problems by raising their lower bound so as to match their upper bound. Some cases are still open that we are confident to solve with similar techniques.

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