Contiguous Allocation of Binary Valued Indivisible Items on a Path

Extended Abstract

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ABSTRACT

We study the problem of allocating indivisible binary-valued items on a path among agents. The objective is to find a fair and efficient allocation in which each agent's bundle forms a contiguous block on the path. We demonstrate that deciding whether every item can be allocated to an agent who wants it is NP-complete. Consequently, we provide fixed-parameter tractable (FPT) algorithms for maximizing utilitarian social welfare, with respect to the optimum value and the number of agents. Additionally, we present a 2-approximation algorithm for the special case when the maximum utility is equal to the number of items. Furthermore, we establish that deciding whether the maximum egalitarian social welfare is at least 2 or at most 1 is an NP-complete problem. We also explore the case where the order of the blocks of items allocated to the agents is predetermined. In this case, we show that both maximum utilitarian social welfare and egalitarian social welfare can be computed in polynomial time. However, we determine that checking the existence of an EF1 allocation is NP-complete.

KEYWORDS

Cake cutting; Contiguous allocation; Envy-freeness

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1 INTRODUCTION

Imagine a scenario in which multiple organizers wish to use the same conference center for their events. Each organizer has a preferred schedule for their events. Typically, organizers prefer to schedule their events in contiguous blocks of time rather than splitting them into separate periods. This leads to the following question: How should the conference center committee schedule time in a contiguous block of time for the different organizers?

A fundamental task in such allocation task is to achieve both fairness and efficiency. Fair division is one of the most fundamental and well-studied topics in computational social choice theory [2, 13, 17] and has received significant attention in the domains of mathematics, economics, political science, and computer science [12, 26, 30, 31, 34]. Fair division problems are of particular interest because of their various real-world applications, such as students sharing the cost of renting an apartment, spouses sharing assets after divorce, and nations claiming ownership of disputed territories. Research discussions on fair division often explore between two distinct categories of items. Certain items, such as cake and land, are considered divisible due to their ability to be divided among agents in an arbitrary manner [5–7, 14, 16, 18, 27]. Additional items, such as residences and automobiles, possess indivisible characteristics, necessitating their allocation in their whole to a single agent [4, 8, 10, 20, 24, 25, 28, 29, 33]. This paper deals with indivisible items. For example, in the scheduling scenario, we consider the case where time slots (e.g., 10-minute increments) are provided in advance.

A natural criterion for assessing the quality of an allocation is *utilitarian social welfare*, which is defined as the sum of the utilities among all agents. Another criterion is *egalitarian social welfare*, which is defined as the minimum of the utilities of all agents. One of the most prominent fairness notions is *envy-freeness (EF)*, which means that no agent envies another based on the sets of items that they receive. Since EF is a strong fairness guarantee, there are also its relaxations to consider. One standard such relaxation is *envy-free up to one item (EF1)*, which requires that any envy that one agent has toward another can be eliminated by removing one item from the envied agent's bundle. Other fairness criteria include *maximin share guarantee (MMS)*, *proportionality (PROP)*, and *equitability (EQ)*. The formal definition of these criteria will be provided in Section 2.

This paper explores the division of items that are arranged in a path while imposing the restriction that only contiguous subsets of items can be assigned to the agents. Our primary focus is on scenarios where each agent employs a binary valuation function, as this represents the most fundamental and crucial setting. We investigate the computational complexities of finding a contiguous allocation that meets a specified fairness and efficiency criterion. Furthermore, we also examine a constraint where the blocks are

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assigned to agents in a specific order of agents. In the scheduling scenario, this constraint means that the ordering of events is predetermined.

2 PRELIMINARIES

We study the problem of allocating indivisible items where the items are arranged on a path, and each agent has a binary valuation for each item. We call this setting *contiguous-allocation of binary-additive items on a path (CBP)*. For a positive integer k, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. Let $M = \{g_1, g_2, \ldots, g_m\}$ denote the set of m indivisible items, and $N = [n]$ be the set of agents. Assume that items are aligned on a path in the order of indices. Each agent $i \in N$ has a binary additive valuation $v_i \colon 2^M \to \mathbb{Z}$, where $v_i(\{g\}) \in \{0, 1\}$ for each $g \in M$ and $v_i(X) = \sum_{g \in X} v_i(\{g\})$ for each $X \subseteq M$. An instance of CBP is $(N, M, (v_i)_{i \in N})$.

An allocation $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ is a partition of all items into bundles for the agents, meaning that agent i receives bundle A_i. We call an allocation $\mathbf{A}$ *contiguous* if each bundle A_i forms a contiguous block of items on the path, i.e., $A_i = \{g_k, g_{k+1}, \ldots, g_\ell\}$ for some k and ℓ. Moreover, a contiguous allocation $\mathbf{A}$ is called *order-consistent* if the blocks are assigned to agents in a specific order, i.e., there exist indices $1 = k_1 \leq k_2 \leq \cdots \leq k_n \leq k_{n+1} = m+1$ such that $A_i = \{g_{k_i}, g_{k_i+1}, \ldots, g_{k_{i+1}-1}\}$ for each $i \in N$. We consider two settings: the fixed-order setting and the flexible-order setting. In the fixed-order setting, we only allow contiguous allocations that are order-consistent. In the flexible-order setting, we allow all the contiguous allocations.

An allocation $\mathbf{A}$ is called *envy-free* if, for all $i, j \in N$, it holds that $v_i(A_i) \geq v_i(A_j)$. In addition, an allocation $\mathbf{A}$ is called *envy-free up to one item (EF1)* if, for all $i, j \in N$, it holds that $v_i(A_i) \geq v_i(A_j \setminus X)$ for some $X \subseteq A_j$ with $|X| \leq 1$. The *utilitarian social welfare* and the *egalitarian social welfare* of an allocation $\mathbf{A}$ are defined as $\sum_{i \in N} v_i(A_i)$ and $\min_{i \in N} v_i(A_i)$, respectively. We call an allocation $\mathbf{A}$ is *U-max* and *E-max* if it maximizes utilitarian social welfare and egalitarian social welfare, respectively. An allocation $\mathbf{A}$ is called *Pareto-optimal (PO)* if, for any other allocation $\mathbf{A}'$, we have $v_i(A_i) = v_i(A'_i)$ for all $i \in N$ or $v_i(A_i) > v_i(A'_i)$ for some $i \in N$. Clearly, any U-max allocation is PO. The maximin share guarantee of an agent i is defined as $\mathrm{MMS}(i) = \max_{\mathbf{A} \in \mathcal{A}} \min_{j \in N} v_i(A_j)$, where $\mathcal{A}$ is the set of all possible contiguous allocations. An allocation $\mathbf{A}$ is said to be *maximin share (MMS)* if $v_i(A_i) \geq \mathrm{MMS}(i)$ for every $i \in N$. Moreover, an allocation $\mathbf{A}$ is said to be *proportional (PROP)* and *equitable (EQ)* if $v_i(A_i) \geq v_i(M)/n$ $(\forall i \in N)$ and $v_i(A_i) = v_j(A_j)$ $(\forall i, j \in N)$, respectively.

3 RESULTS

In this section, we present our results, which are summarized as Table 1. We refer the reader to the full version of our paper [23] for the proofs.

We provide the following algorithmic results.

Theorem 3.1. *The U-max problem of CBP can be solved in $O(mn)$ time for the fixed-order setting and is FPT with respect to the number of agents for the flexible-order setting.*[1]

Theorem 3.2. *For the fixed-order setting, the EQ problem of CBP can be solved in $O(m^2(m + n))$ time.*

Table 1: The computational complexities of checking the existence of an allocation of binary-valued items that satisfies a designated property and constructing one, if it exists

	contiguous		unconstrained
	flexible-order	fixed-order	
EF	NP-h [19]	NP-h [19]	NP-h [3]
EF1	open	**NP-h (Thm. 3.7)**	P [15, 29]
U-max	**NP-h (Thm. 3.8)**	**P (Thm. 3.1)**	P†
E-max	**NP-h (Thm. 3.9)**	**P (Thm. 3.3)**	P‡
PO	P [22]	**P (Thm. 3.1)**	P†
MMS	P [9]	**P (Thm. 3.5)**	P [11]
PROP	NP-h [19]	**P (Thm. 3.4)**	P‡
EQ	NP-h [19]	**P (Thm. 3.2)**	P‡

† These can be solved by just allocating each item to an arbitrary agent who values it.
‡ These can be solved by a max-flow algorithm (see, e.g., [32]).

Theorem 3.3. *The E-max problem of CBP can be solved in $O((m+n)\log m)$ time for the fixed-order setting and is FPT with respect to the number of agents for the flexible-order setting,*

Theorem 3.4. *For the fixed-order setting, the PROP problem of CBP can be solved in $O(mn)$ time.*

Theorem 3.5. *For the fixed-order setting, the MMS problem of CBP can be solved in $O(mn \log m)$ time.*

Theorem 3.6. *For the flexible-order setting, the U-max problem of CBP admits a $1/2$-approximation algorithm and is FPT with respect to the optimal value.*

Moreover, we provide the following hardness results.

Theorem 3.7. *For the fixed-order setting, the EF1 problem of CBP is NP-hard.*

Theorem 3.8. *For the flexible-order setting, the U-max problem of CBP is NP-hard. Moreover, it is NP-complete to determine whether the optimal utilitarian social welfare is equal to the number of items.*

Theorem 3.9. *For the flexible-order setting, the E-max problem of CBP is NP-hard. Specifically, it is NP-complete to determine whether the optimal egalitarian social welfare is at least 2 or at most 1.*

4 CONCLUDING REMARKS

A straightforward future work is to construct a faster parameterized algorithm for maximizing utilitarian social welfare. There is also the scope of finding better approximation algorithms or better inapproximability of the U-max and E-max problems. Specifically, it is open whether the $1/8$-approximation algorithm of Aumann et al. [1] for the U-max problem with additive valuations can be improved in the special case of binary additive. Igarashi [21] proved that an EF1 allocation always exists and posed a question regarding the computational complexity of finding it for the flexible-order setting. While we show that finding such an allocation is NP-hard for the fixed order case, the complexity of the problem for the general flexible order setting remains open.

[1] A faster FPT was given by Aumann et al. [1].

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