The Degree of (Extended) Justified Representation and Its Optimization*

Biaoshuai Tao
Shanghai Jiao Tong University
Shanghai, China
bstao@sjtu.edu.cn

Chengkai Zhang
Shanghai Jiao Tong University
Shanghai, China
zhangchengkai@sjtu.edu.cn

Houyu Zhou†
UNSW Sydney
Sydney, Australia
houyu.zhou@unsw.edu.au

ABSTRACT

Justified Representation (JR)/Extended Justified Representation (EJR) is a desirable axiom in multiwinner approval voting. In contrast to that (E)JR only requires at least *one* voter to be represented in every cohesive group, we study its optimization version that maximizes the *number* of represented voters in each group. Given an instance, we say a winning committee provides a JR degree (EJR degree, resp.) of c if at least c voters in each ℓ-cohesive group (1-cohesive group, resp.) have approved ℓ (1, resp.) winning candidates. Hence, every (E)JR committee provides the (E)JR degree of at least 1. Besides proposing this new property, we propose the optimization problem of finding a winning committee that achieves the maximum possible (E)JR degree, called MDJR and MDEJR, corresponding to JR and EJR respectively.

We study the computational complexity and approximability of MDJR of MDEJR. An (E)JR committee, which can be found in polynomial time, straightforwardly gives a (k/n)-approximation. We also show that the original algorithms for finding a JR and an EJR winner committee are also $1/k$ and $1/(k+1)$ approximation algorithms for MDJR and MDEJR respectively. On the other hand, we show that it is NP-hard to approximate MDJR and MDEJR to within a factor of $(k/n)^{1-\epsilon}$ and to within a factor of $(1/k)^{1-\varepsilon}$, for any $\epsilon > 0$, which complements the positive results. Next, we study the fixed-parameter-tractability of this problem. We show that both problems are W[2]-hard if k, the size of the winning committee, is specified as the parameter. However, when $c_{\max}$, the maximum value of c such that a committee that provides an (E)JR degree of c exists, is additionally given as a parameter, we show that both MDJR and MDEJR are fixed-parameter-tractable.

CCS CONCEPTS

• **Theory of computation** → **Algorithmic game theory**; • **Applied computing** → *Economics*; *Voting / election technologies*; • **Computing methodologies** → *Multi-agent systems.*

KEYWORDS

Justified Representation; Multi-winner Approval Voting; (E)JR Degree

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†Corresponding Author.

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1 INTRODUCTION

An approval-based committee voting rule (ABC rule) plays a crucial role in collective decision-making by determining a committee from a set of m candidates $C = \{c_1, \ldots, c_m\}$, a set of n voters $N = \{1, \ldots, n\}$ who each approve a subset A_i of C, and an integer k representing the desired committee size. Let the list $\mathbf{A} = (A_1, \ldots, A_n)$ of approval ballots be the ballot profile. Formally, ABC rules take a tuple $(N, C, \mathbf{A}, k)$ as input, where k is a positive integer that satisfies $k \leq |C|$, and return one or more size-k subsets $W \subseteq C$, which are called the winning committees. In many concrete voting systems a tiebreaking method is included so that a resolute outcome is guaranteed. In some cases, N and C are omitted from the notation when they are clear from the context. ABC rules are widely applied in various contexts, including the election of representative bodies (such as supervisory boards and trade unions), identifying responses to database queries [11, 18, 33], selecting validators in consensus protocols like blockchain [10], making collective recommendations for groups [23, 24], and facilitating discussions on proposals within liquid democracy [5]. Additionally, as committee elections fall under the domain of participatory budgeting (PB) [9, 17], a strong grasp of ABC rules is indispensable for designing effective PB methods.

The applicability of various ABC rules often depends on the particular context, yet an important requirement one often imposes on an ABC rule is that it can accurately reflect the voters' preferences, e.g., every large group of voters should justify a seat in the committee. In recent years, a desirable axiom has been proposed and developed, *Justified Representation* (JR), which requires every group of at least n/k voters that have at least one common candidate should be represented.

Given a profile $\mathbf{A}$ of n approval preferences, a subgroup of voters, denoted by $N' \subseteq N$ is called an ℓ-cohesive group for some $\ell \in \mathbb{N}$, if $|N'| \geq \ell \cdot \frac{n}{k}$ and $|\bigcap_{i \in N'} A_i| \geq \ell$. If $\ell = 1$, we say it is a cohesive group for short.

Definition 1 (Justified representation (JR)). *[2]* Given a ballot profile $\mathbf{A} = (A_1, \ldots, A_n)$ over a candidate set C and a committee size k, we say that a set of candidates W of size $|W| = k$ satisfies JR for $(\mathbf{A}, k)$ if, for every cohesive group (defined right above), there is at least one voter which approves at least one candidate in W.

We say that an ABC rule satisfies JR if for each profile $\mathbf{A}$ and committee size k, each winning committee provide JR.

Many well-known ABC rules have been shown to satisfy JR, from a simple greedy approval voting rule to a more complex proportional approval voting (PAV) rule. Hence, some stronger JR axioms are proposed to distinguish the existing approval voting rules, such as *Extended Justified Representation* (EJR). EJR requires that in every ℓ-cohesive group, at least one member is represented. Hence, EJR implies JR.

Definition 2 (Extended Justified Representation). *[2]* Given a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ over a candidate set C and a committee size k, we say that a set of candidates W of size $|W| = k$ satisfies EJR for $(\mathbf{A}, k)$ if, for every ℓ-cohesive group with every $\ell \in [k]$, there is at least one voter which approves at least ℓ candidates in W.

We say that an ABC rule satisfies EJR if for each profile $\mathbf{A}$ and committee size k, each winning committee provide EJR.

While there are a wealth of concepts associated with JR, there are instances where comparing the performance of different winning committees becomes impossible. To illustrate this situation, consider a brief example.

Example 1. *Let $k = 1$, $N = \{1, 2, 3, 4\}$, $C = \{c_1, c_2, c_3, c_4\}$, $A_1 = \{c_1\}$, and $A_i = A_{i-1} \cup \{c_i\}$ for every $i = 2, 3, 4$. We are interested in four different committees $W_i = \{c_i\}$ for every $i \in [4]$.*

Since Example 1 is a single winner voting instance, we have $k = \ell = 1$. Clearly, N is the only cohesive group in this example, and all four committees in Example 1 satisfy all the above JR axioms: besides the JR and EJR axioms mentioned earlier, one can verify that those committees also satisfy some other JR axioms such as FJR [30], PJR+, and EJR+ [8]. However, W_{i-1} is intuitively better than W_i since it satisfies more voters. To distinguish the performance of these committees, besides proposing new JR variants, another approach to addressing this scenario involves leveraging *quantitative* techniques to model a hierarchy of JR, which has a similar spirit to the work of using quantitative techniques to model proportionality.

Definition 3 (Proportionality degree). [32] Fix a function $f : \mathbb{N} \to \mathbb{R}$. An ABC rule has a *proportionality degree* of f if for each instance (A, k), each winning committee W, and each ℓ-cohesive group V, the average number of winners that voters from V approve is at least $f(\ell)$, i.e.,

$$\frac{1}{|V|} \sum_{i \in V} |A_i \cap W| \geq f(\ell).$$

Since $k = 1$ in Example 1, it is not hard to see that W_1, W_2, W_3, W_4 achieve the proportionality degree of 1, 3/4, 1/2, 1/4, respectively.

Within each cohesive group, the proportionality degree measures the *average satisfaction*, which may not align with the *number of represented voters*. Specifically, for a cohesive group where not all voters are represented, optimizing the proportionality degree may further increase the satisfaction of voters who are already represented while leaving some other voters completely unrepresented. The following example demonstrates this.

Example 2. *Let $k = 3$, $N = \{1, 2, \dots, 9\}$, and $C = \{c_1, c_2, \dots, c_6\}$, where*

- *c_1 is approved by voters 1, 2, 3, 4, 5;*
- *c_2 is approved by voters 4, 5, 6, 7, 8;*
- *c_3 is approved by voters 7, 8, 9, 1, 2;*
- *each of the candidates c_4, c_5, c_6 is approved by voters 1, 2, 4, 5, 7, 8.*

In this example, the winner committee that optimizes the proportionality degree is $\{c_4, c_5, c_6\}$, where we have $f(1) = 2$ and $f(2) = 3$, and the value of f is maximized at both 1 and 2 (where f is the function defined in Definition 3 but with a fixed instance). In this case, voters 3, 6, 9 are completely unsatisfied.

On the other hand, under the winner committee $\{c_1, c_2, c_3\}$, every voter is represented in both JR and EJR senses. To see this, all voters approve at least one winner, so all voters in each 1-cohesive group are represented; there is only one 2-cohesive group $\{1, 2, 4, 5, 7, 8\}$, and each voter in this cohesive group approves two winners. However, this winner committee is sub-optimal in the proportionality degree measurement: $f(1) = 5/3$ and $f(2) = 2$.

In the example above, the winner committee $\{c_4, c_5, c_6\}$ is clearly "unfair" to voters 3, 6, 9, and the winner committee $\{c_1, c_2, c_3\}$ is much more appealing, at least in some applications. Indeed, many social choice scenarios require the committee to represent as many voters as possible (e.g., Monroe's rule and the social coverage objective) rather than to increase the average satisfaction. Hence, while maximizing average satisfactions of cohesive groups measured by the proportionality degree is a natural goal in some applications, in some other scenarios, increasing the number of represented voters is a more appealing goal. Motivated by these, we seek to define a new quantitative notion that describes the *number of represented voters* (instead of the average satisfaction) in each cohesive group.

There is also a natural follow-up question: how can we find the committee that satisfies the optimal (maximum) justified representation degree, given an instance $(N, C, \mathbf{A}, k)$? In this paper, we propose the degree of (E)JR and study its optimization problem.

1.1 Our New Notions

In this paper, we study the degree of (E)JR and its optimization problem.

Proposal of new notion—(E)JR degree. Intuitively, given a ballot instance, we say a winning committee provides *an (E)JR degree of c* if at least c voters in every cohesive group are represented.

Definition 4 (JR Degree). Given a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ over a candidate set C and a committee size k, we say that a set of candidates W of size $|W| = k$ *achieves JR degree c* for $(\mathbf{A}, k)$ if, for every cohesive group, there are at least c voters each of which approves at least 1 candidate in W.

Definition 5 (EJR degree). Given a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ over a candidate set C and a committee size k, we say that a set of candidates W of size $|W| = k$ *achieves EJR degree c* for $(\mathbf{A}, k)$ if, for every ℓ-cohesive group with every $\ell \in [k]$, there are at least c voters each of which approves at least ℓ candidates in W.

In this paper, we only consider ballot instances where at least one cohesive group exists, to avoid the uninteresting degenerated case (with no cohesive group) that invalidates the above two definitions.

From the above definitions, a winner committee satisfying JR has a JR degree of at least 1, and a winner committee satisfying EJR has an EJR degree of at least 1. In Example 1, W_1, W_2, W_3, W_4 achieve the (E)JR degree of 4,3,2,1, respectively.

Relationship between JR degree and EJR degree. In general, given any ballot instance with n voters and a winner committee of size k, one can see that both the JR degree and the EJR degree are at most $\lceil \frac{n}{k} \rceil$: if a cohesive group exists, there is always a cohesive group with size exactly $\lceil \frac{n}{k} \rceil$. On the other hand, it is widely known that an EJR committee always exists (and so does a JR committee), and many algorithms are known to find an EJR committee [2, 12] some of which run in polynomial time [3, 8, 30]. Therefore, for any ballot instance, the maximum degrees for both JR and EJR are at least 1 and at most $\lceil \frac{n}{k} \rceil$.

Since EJR implies JR, it is easy to see from our definitions that, if a winner committee provides an EJR degree of c, its JR degree is at least c. However, a winner committee may have a higher JR degree than its EJR degree. In addition, the winner committee that maximizes the EJR degree may not be the same as the winner committee that maximizes the JR degree. The following proposition, whose proof is deferred to the full version of our paper, shows that the difference between the JR degree and the EJR degree can be significant.

Proposition 1. *For any $\gamma > 0$, there is a ballot instance with maximum JR degree c^*_{JR} and maximum EJR degree c^*_{EJR} such that*

(1) $c^*_{JR}/c^*_{EJR} > \gamma$,
(2) *all winner committees with EJR degree c^*_{EJR} have JR degrees at most c^*_{JR}/γ, and*
(3) *all winner committees with JR degree c^*_{JR} have EJR degrees at most c^*_{EJR}/γ.*

While a committee satisfying EJR is generally considered better than one that only satisfies JR, the JR degree and EJR degree are not directly comparable. Intuitively, the EJR degree faces more challenge of satisfying voters in larger cohesive groups, i.e., ℓ approved winners needed for representing a voter in a ℓ-cohesive group, whereas in the JR degree setting, a voter is represented if there is an approved winner. Depending on the scenario, different degree measurements may be more appropriate. Thus, both metrics are valuable and merit further study.

On optimizing (E)JR degree. The definition of (E)JR degree naturally motivates the following optimization problem, which we define as MDJR/MDEJR.

Definition 6 (Maximum (E)JR degree, MDJR (MDEJR))**.** Given an instance (A, k), MDJR (MDEJR) outputs one committee that achieves the maximum (E)JR degree.

"Number" versus "fraction". We have defined the JR and EJR degrees based on the *number* of represented voters. Another natural way is to define both notions based on the *fraction* of represented voters.

When dealing with the JR degree, both definitions are equivalent. To see this, in terms of both numbers and fractions, the least satisfying cohesive group always contains exactly $\lceil n/k \rceil$ voters, as removing a represented voter from a cohesive group of more than $\lceil n/k \rceil$ voters would make both the number and the fraction of represented voters decrease. Given that we are concerning the least satisfying cohesive group (which always has the same size), the "number version" and the "fraction version" of the JR degree are equivalent.

When dealing with the EJR degree, the two definitions are different. The "number version" fits better with the spirit of EJR. In the definition of EJR (see Definition 2), the requirement is that *one* voter needs to be represented (approve at least ℓ winners) in every ℓ-cohesive group, instead of being that ℓ voters need to be represented in every ℓ-cohesive group. On the other hand, for some $\ell > 1$, given an ℓ-cohesive group with minimum size $\ell \cdot \frac{n}{k}$ and a 1-cohesive group with minimum size $\frac{n}{k}$, if one voter is represented in the 1-cohesive group, then ℓ voters in the ℓ-cohesive group need to be represented in order to make the ℓ-cohesive as "happy" as the 1-cohesive group in the case the EJR degree is defined in the "fraction version".

1.2 Our Technical Contributions

In this paper, we focus on the computational complexity and the approximability of the optimization problems MDJR and MDEJR. Our results are listed below.

We first show that the algorithm for finding a JR committee proposed by Aziz et al. [2] also provides a $\frac{1}{k}$-approximation to MDJR, and the algorithm for finding an EJR committee proposed by Aziz et al. [3] also provides a $\frac{1}{k+1}$-approximation to MDEJR. On the other hand, since the maximum (E)JR degree for both MDJR and MDEJR is $\lceil \frac{n}{k} \rceil$, the approximation guarantees for the two algorithms above can also be written as $1/\lceil \frac{n}{k} \rceil \approx k/n$. To complement these positive results, we show almost tight inapproximability results. We show that it is NP-hard to approximate MDJR (MDEJR) within a factor of $(k/n)^{1-\epsilon}$ for any $\epsilon > 0$. We also show that it is NP-hard to approximate MDJR (MDEJR) within a factor of $(1/k)^{1-\epsilon}$ for any $\epsilon > 0$.

We study the fixed-parameter-tractability of this problem. We show that finding a committee with the maximum achievable (E)JR degree is W[2]-hard if k, the size of the winning committee, is specified as the parameter.

When the maximum achievable (E)JR degree of an instance is additionally given as a parameter, we show that the problem is fixed-parameter-tractable.

Surprisingly, although Proposition 1 demonstrates that JR degree and EJR degree have different natures, we obtain the same set of results for MDJR and MDEJR.

1.3 Further Related Work

In this subsection, we discuss the related work on justified representation (JR). In addition to the JR axioms previously mentioned, several other JR-related axioms have been proposed and studied. Fernández et al. [15] introduced Proportional JR (PJR), which requires that every ℓ-cohesive group have some ℓ winners represented in the union of their approval sets. PJR is weaker than Extended JR (EJR) but stronger than JR. The authors also proposed Perfect Representation (PR), aiming to represent all voters by some winners, with each winner representing $\frac{n}{k}$ voters. While a PR committee may not always exist, if one does, it can be verified that such a committee guarantees the maximum JR degree of $\frac{n}{k}$.

Peters et al. [30] introduced Fully JR (FJR), which weakens the cohesiveness requirement. It considers groups of $\ell\frac{n}{k}$ voters who share at least $\beta \leq \ell$ candidates in common. A committee satisfies FJR if every (ℓ, β)-weak-cohesive group, where $\ell \in [k]$ and $\beta \in [\ell]$, has at least one member who approves β winners. Notably, FJR implies EJR. Brill and Peters [8] proposed PJR+ and EJR+, which focus on ensuring PJR and EJR for groups of $\ell\frac{n}{k}$ voters, where at least one non-elected candidate is approved by all group members, as opposed to considering only ℓ-cohesive groups. Consequently, EJR+ implies both EJR and PJR+, while PJR+ implies PJR.

Brill et al. [7] studied Individual Representation (IR), which requires that every voter in an ℓ-cohesive group is represented. An IR committee would achieve the maximum (E)JR degree, although such committees may not always exist. The potential non-existence of IR and PR committees, compared to the guaranteed existence of (E)JR committees, motivates us to explore a quantitative measure bridging (E)JR, PR, and IR.

Moreover, JR has been investigated in other domains, such as fair division [6, 25], participatory budgeting [4, 30], and facility location games [14]. Other properties that assess a committee's proportionality, such as laminar proportionality and priceability, have also been considered [31].

In our work, we study voting rules that maximize the (E)JR degree, particularly MDJR and MDEJR. In multi-winner voting, there are a variety of voting rules that maximize certain scores, collectively known as Thiele methods [34]. Thiele methods focus on maximizing the sum of voters' individual satisfaction, where a voter's satisfaction is determined by the number of approved candidates in the winning committee. While our MDEJR maximization draws inspiration from this concept, MDJR maximization fundamentally differs, as it focuses on maximizing the JR degree within each cohesive group rather than global satisfaction.

Monroe's rule [27] shares a similar objective with our MDJR approach, as it seeks to maximize the number of voters represented by at least one candidate in the winning committee. This mirrors our goal of maximizing the number of voters represented by at least one winner in every cohesive group. Other notable voting rules, such as Phragmén's rules [19], have also been studied in the context of multi-winner voting [22].

Finally, several well-known properties in multi-winner approval voting are relevant to our study, including anonymity and neutrality [1, 26, 28], Pareto efficiency [20], monotonicity [16], consistency [21], and strategyproofness [29].

2 APPROXIMABILITY OF MDJR AND MDEJR

In this section, we study the approximability of MDJR and MDEJR. We provide almost tight approximability of both problems, in terms of both k and k/n.

As we mentioned before, the maximum degree for both MDJR and MDEJR is $\lceil \frac{n}{k} \rceil$. On the other hand, any winner committee satisfying JR (EJR, resp.) gives a JR degree (EJR degree, resp.) of at least 1. Therefore, the algorithm for finding a JR (EJR, resp.) committee provides a $1/\lceil \frac{n}{k} \rceil \approx \frac{k}{n}$ approximation to MDJR (MDEJR, resp.). In Sect. 2.1, we will show that the algorithm for finding a JR committee proposed by Aziz et al. [2] also provides a $\frac{1}{k}$-approximation to MDJR, and the algorithm for finding an EJR committee proposed by Aziz et al. [3] also provides a $\frac{1}{k+1}$-approximation to MDEJR.

In Sect. 2.2, we show that the approximation ratios of approximately k/n and $1/k$ mentioned above are almost tight.

2.1 Approximation Algorithms

The algorithm GreedyAV (Algorithm 1) proposed by Aziz et al. [2] always outputs a JR committee. We will show in Theorem 1 that it provides a $\frac{1}{k}$-approximation to MDJR.

Algorithm 1: Greedy Approval Voting (GreedyAV)

Input: An instance $\mathcal{I} = (N, C, \mathbf{A}, k))$
Output: A winning committee W of size k

```
W ← ∅
for j ∈ [k] do
    Let c be the candidate approved by the maximum
      number of voters in N
    W ← W ∪ c
    N ← N \ V where V is the set of voters who approve c
return W
```

We prove the following proposition first.

Proposition 2. *GreedyAV outputs a committee achieving JR degree at least $\frac{n}{k^2}$.*

Proof. We prove it by contradiction. Suppose the committee output by GreedyAV provides the JR degree less than $\frac{n}{k^2}$. Hence, there will be at least $(\frac{n}{k} - \frac{n}{k^2} + 1)$ voters approving no candidate in W that approve a common candidate $c \notin W$. From the definition of GreedyAV, the coverage of voters is at least $(\frac{n}{k} - \frac{n}{k^2} + 1)$ in each iteration, and at least $\frac{n}{k}$ in the first iteration (given that there is at least one cohesive group). Therefore, the total number of voters is at least

$$\begin{aligned} &\frac{n}{k} + (k-1)\left(\frac{n}{k} - \frac{n}{k^2} + 1\right) + \left(\frac{n}{k} - \frac{n}{k^2} + 1\right) \\ =&n + \frac{n}{k} - k\left(\frac{n}{k^2} - 1\right) > n, \end{aligned}$$

which leads to a contradiction in the number of voters. □

Now, we are ready to show the approximation guarantee of GreedyAV.

Theorem 1. *GreedyAV runs in polynomial time and provides a $\frac{1}{k}$-approximation to* MDJR.

Proof. It is clear that the algorithm runs in polynomial time. The minimum JR degree of $\frac{n}{k^2}$ proved in Proposition 2 is sufficient to guarantee the $\frac{1}{k}$-approximation, as we have remarked that the maximum possible JR degree is $\lceil \frac{n}{k} \rceil$. Detailed treatments for the ceiling function are available in the full version of this paper. □

Similar to MDJR, we first consider an EJR voting rule, *proportional approval voting* (PAV) [2], which outputs the committee that maximizes the PAV-score, where the PAV-score of a committee $W \subseteq C$ is defined as

$$s_{\text{PAV}}(W) = \sum_{i=1}^{n} \sum_{j=1}^{|A_i \cap W|} \frac{1}{j}.$$

In the PAV-score, each voter's "utility" is defined by the harmonic progression $H[t]$ for t approved winning candidates and the PAV-score can then be understood as the *social welfare.* One may wonder whether PAV can provide the maximum EJR degree. We find that

PAV fails to achieve the maximum EJR degree in some instances. A counterexample can be found in the full version of our paper.

In addition, PAV cannot be computed in polynomial time. However, PAV cannot be computed in polynomial time. Aziz et al. [3] showed that a local search alternative algorithm for PAV can both satisfy EJR and be computed in polynomial time. The algorithm is described in Algorithm 2. Starting from an arbitrary winning committee, the algorithm considers all possible single-candidate-replacements that increase the PAV score by at least λ (where λ is a parameter of the algorithm). For each pair of candidate c^+ and c^- with $c^+ \notin W$ and $c^- \in W$, if we swap c^+ and c^-, i.e. to remove c^- from the committee and select c^+ instead, the score is increased by $\Delta(W, c^+, c^-) = s_{\text{PAV}}(W \setminus \{c^-\} \cup \{c^+\}) - s_{\text{PAV}}(W)$.

Algorithm 2: λ-LS-PAV

Input: An instance $\mathcal{I} = (N, C, \mathbf{A}, k))$
Output: A winning committee W of size k

```
1 W ← k arbitrary candidates from C
2 while there exist c+ ∉ W and c− ∈ W such that
    Δ(W, c+, c−) ≥ λ do
3   W ← W \ {c−} ∪ {c+}
4 return W
```

Next, we will show that λ-LS-PAV runs in polynomial time and provides a $\frac{1}{k+1}$-approximation to MDEJR for a suitable choice of λ.

We show the following proposition first.

Proposition 3. *λ-LS-PAV outputs a committee achieving EJR degree at least $c^* = \frac{n}{k(k+1)} - \lambda \frac{k}{k+1}$ for any $\lambda \in [0, \frac{n}{k^2})$.*

Proof. We prove it by contradiction. Suppose the committee, W, output by λ-LS-PAV provides the EJR degree strictly smaller than c^*. There exists a ℓ-cohesive group $V \subseteq N$, such that less than c^* voters in V approves at least ℓ members of W, i.e., $|\{i \in V : |A_i \cap W| \geq \ell\}| < c^*$. Since V is ℓ-cohesive, there exist ℓ candidates approved by all voters in V. At least one such candidate, $c^+ \in \bigcap_{i \in V} A_i$, is not selected, as otherwise all voters in V approve at least ℓ members of W.

We will show that there exists a candidate $c^- \in W$ such that the increment of the score by swapping c^+ and c^- is at least λ, so W cannot be an output of λ-LS-PAV, leading to a contradiction. To see this, we try to swap c^+ and any candidate $c^- \in W$. Since $c^+ \in A_i$ for all $i \in V$, we have

$$\begin{aligned}\Delta(W, c^+, c^-) &= \sum_{\substack{i: c^+ \in A_i \\ c^- \notin A_i}} \frac{1}{|A_i \cap W| + 1} - \sum_{\substack{i: c^+ \notin A_i \\ c^- \in A_i}} \frac{1}{|A_i \cap W|} \\ &\geq \sum_{i \in V : c^- \notin A_i} \frac{1}{|A_i \cap W| + 1} - \sum_{i \in N \setminus V : c^- \in A_i} \frac{1}{|A_i \cap W|},\end{aligned}$$

and, by summing up $\Delta(W, c^+, c^-)$ for $c^- \in W$,

$$\begin{aligned}&\sum_{c^- \in W} \Delta(W, c^+, c^-) \\ &\geq \sum_{c^- \in W} \sum_{\substack{i \in V: \\ c^- \notin A_i}} \frac{1}{|A_i \cap W| + 1} - \sum_{c^- \in W} \sum_{\substack{i \in N \setminus V: \\ c^- \in A_i}} \frac{1}{|A_i \cap W|} \\ &= \sum_{i \in V} \sum_{\substack{c^- : c^- \in W \\ c^- \notin A_i}} \frac{1}{|A_i \cap W| + 1} - \sum_{i \in N \setminus V} \sum_{\substack{c^- : c^- \in W \\ c^- \in A_i}} \frac{1}{|A_i \cap W|} \\ &= \left(\sum_{i \in V} \frac{k - |A_i \cap W|}{|A_i \cap W| + 1} \right) - (n - |V|) \\ &= \left(\sum_{i \in V} \frac{k + 1}{|A_i \cap W| + 1} \right) - n \\ &\geq \left(\sum_{i \in V : |A_i \cap W| < \ell} \frac{k + 1}{|A_i \cap W| + 1} \right) - n \\ &\geq \left(\sum_{i \in V : |A_i \cap W| < \ell} \frac{k + 1}{\ell} \right) - n.\end{aligned}$$

Since less than c^* voters in V approve at least ℓ members of W, the number of voters in V approve less than ℓ members of W is at least $|V| - c^* + 1$. Thus,

$$\begin{aligned}\sum_{c^- \in W} \Delta(W, c^+, c^-) &\geq \left(\sum_{i \in V : |A_i \cap W| < \ell} \frac{k + 1}{\ell} \right) - n \\ &\geq (k + 1) \frac{|V| - c^* + 1}{\ell} - n \\ &> (k + 1) \left(\frac{|V|}{\ell} - c^* \right) - n \\ &\geq (k + 1) \left(\frac{n}{k} - c^* \right) - n \\ &= \frac{n}{k} - (k + 1) c^* \\ &= \lambda k.\end{aligned}$$

From the pigeonhole principle, it follows that there exists a candidate $c^- \in W$ such that $\Delta(W, c^+, c^-) \geq \lambda$. □

Now, we are ready to conclude the approximation guarantee of λ-LS-PAV.

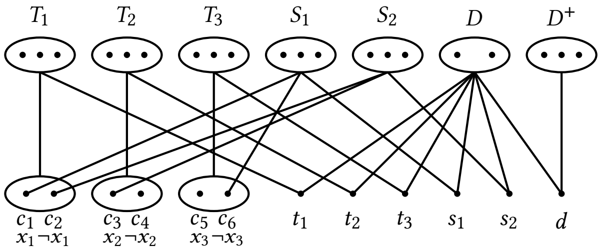
Theorem 2. *For $\lambda = \frac{1}{2k^2}$, λ-LS-PAV runs in polynomial time and provides a $\frac{1}{k+1}$-approximation to* MDEJR.

Proof. This follows from taking $\lambda = \frac{1}{2k^2}$ in Proposition 3. Details of this proof are available in the full version of our paper. □

2.2 Matching Inapproximability Results

To complement the positive results mentioned in the previous section, we present the following inapproximability results.

Theorem 3. *It is NP-hard to approximate both* MDJR *and* MDEJR *within a factor of $\left(\frac{k}{n}\right)^{1-\epsilon}$ for any $\epsilon > 0$.*



$\phi = (x_1 \vee \neg x_3) \wedge (\neg x_1 \vee x_2)$, $\bar{m} = 3$, $\bar{n} = 2$, $k = \bar{m} + 1 = 4$.

Figure 1: An example of the construction with $\phi = (x_1 \vee \neg x_3) \wedge (\neg x_1 \vee x_2)$. The black circles at the top represent voters and the others at the bottom represent candidates. Edges in the graph represent approvals. An edge connecting between a group of voters and a group of candidates indicates that every voter in the voter group approves every candidate in the candidate group.

Theorem 4. *It is NP-hard to approximate* MDJR *and* MDEJR *within a factor of* $\left(\frac{1}{k}\right)^{1-\epsilon}$ *for any* $\epsilon > 0$.

We will simultaneously prove these theorems by constructing a hard ballot instance that is used for all of them. We will make sure the instance we constructed has no ℓ-cohesive group with $\ell > 1$. Notice that the JR degree and the EJR degree for any committee are always the same for instances with only 1-cohesive groups. In addition, we have $n \approx k^2$ in our construction, so the factor k/n is approximately $1/k$.

Before we prove the theorems, we first introduce a NP-hard problem: sparse-SAT problem. One can find variations of SAT problem and a similar argument in Tovey [35].

Definition 7 (sparse-SAT). Given a CNF formula ϕ that, for any pair of variables x and y, at most one clause contains both x (or $\neg x$) and y (or $\neg y$), decide if there is a value assignment to the variables to make ϕ true.

To see its NP-hardness, it can be reduced from the SAT problem. Start with any SAT instance. Without loss of generality, suppose each variable, x or $\neg x$, appears in each clause at most once. For each variable x such that x or $\neg x$ is contained in more than two clauses, we perform the following procedure. Suppose x appears in k clauses. Create k new variables $x_1, \dots, x_k$ and replace the ith occurrence of x with x_i (and $\neg x$ is replaced by $\neg x_i$, respectively) for each $i = 1, \dots, k$. Add the clause $(x_i \vee \neg x_{i+1})$ for $i = 1, \dots, k-1$ and the clause $(x_k \vee \neg x_1)$. Note that, in the new instance, variable x_i and y_j appear in a clause only when the ith occurrence of x and the jth occurrence of y in ϕ are in the same clause, so the new instance satisfies the requirement of the sparse-SAT problem.

In the new instance, the clause $(x_i \vee \neg x_{i+1})$ implies that if x_i is false, x_{i+1} must be false as well. The cyclic structure of the clauses therefore forces the x_i to be either all true or all false, so the new instance is satisfiable if the original one is. Moreover, the transformation requires polynomial time.

Now we are ready to prove our theorems.

Proof of Theorem 3 and 4. We reduce from sparse-SAT problem. Given any sparse-SAT instance ϕ, suppose there are $\bar{n}$ clauses and $\bar{m}$ variables (say $x_1, \dots, x_{\bar{m}}$). We consider an ABC voting instance with $3\bar{m}+\bar{n}+1$ candidates $\{c_1, \dots, c_{2\bar{m}}, s_1, \dots, s_{\bar{n}}, t_1, \dots, t_{\bar{m}}, d\}$ and $\bar{n}\bar{m}+\bar{m}^2+\bar{n}+\bar{m}$ voters. We want to select a committee of size $\bar{m}+1$. Hence, we care about the cohesive group of size $\frac{\bar{n}\bar{m}+\bar{m}^2+\bar{n}+\bar{m}}{\bar{m}+1} = \bar{n}+\bar{m}$. First, for each variable x_j and its negation $\neg x_j$, we create two corresponding candidates c_{2j-1} and c_{2j} and a group T_j of $\bar{m}$ voters who approve c_{2j-1} and c_{2j}. For the ith clause, we create a group S_i of $\bar{m}$ voters. All voters in group S_i approve c_{2j-1} if x_j occurs in the ith clause and c_{2j} if $\neg x_j$ occurs in the ith clause. All voters in group S_i approve s_i and voters in group T_i approve t_i additionally. We create a set D of $\bar{n}$ voters who approve $s_1, \dots, s_{\bar{n}}, t_1, \dots, t_{\bar{m}}, d$. Hence, for each $i \in [\bar{n}]$, $S_i \cup D$ forms a 1-cohesive group, and for each $j \in [\bar{m}]$, $T_j \cup D$ forms a 1-cohesive group. Moreover, we create a set D^+ of $\bar{m}$ voters who approve d. Hence, $D \cup D^+$ forms a 1-cohesive group. An example of our construction is shown in Fig. 1.

Notably, there is no 2-cohesive group. First, candidate s_i for $i \in [\bar{n}]$ or t_j for $j \in [\bar{m}]$ or d has only $\bar{n} + \bar{m}$ voters approving them as constructed above. For candidates $c_1, \dots, c_{2\bar{m}}$, we will show that no $2(\bar{n} + \bar{m})$ voters have two common approved candidates. For any two candidates t_{2i-1}, t_{2i} in $\{c_1, \dots, c_{2\bar{m}}\}$ that correspond to a variable and its negation, the set of voters approving both candidates is exactly T_i (notice that we can assume without loss of generality that x_i and $\neg x_i$ do not appear in the same clause, for otherwise, we can safely remove both literals). We have $|T_i| = \bar{m} < 2(\bar{n} + \bar{m})$. For any two candidates in $\{c_1, \dots, c_{2\bar{m}}\}$ that correspond to different variables, by our sparsity assumption of ϕ, the set of voters approving both candidates is at most some S_j (representing a clause that contains both variables). We have $|S_j| = \bar{m} < 2(\bar{n}+\bar{m})$. Thus, there is no 2-cohesive group or ℓ-cohesive group for $\ell > 1$. In this case, the EJR degree is equal to the JR degree. Thereafter, we will analyze the JR degree and MDJR, and the analysis applies to MDEJR as well.

If there is a value assignment to the variables to make ϕ true, then MDJR will achieve a JR degree of $\bar{n} + \bar{m}$. To see this, for each $j \in [\bar{m}]$, we select c_{2j-1} as a winner if x_j is assigned true, or we select c_{2j}. Then we select d as the winner. We can verify that each voter in $\{S_i\}_{i \in [\bar{n}]} \cup \{T_j\}_{j \in [\bar{m}]}$ approves at least one winner. In addition, d is approved by all voters in D and D^+. Hence, all voters approve at least one winner, and the JR degree equals the size of a 1-cohesive group, which is $\bar{n} + \bar{m}$.

If there does not exist a satisfying assignment to ϕ, then MDJR will achieve the JR degree of at most $\bar{n}$. To see this, we prove it by contradiction. Assume that MDJR can achieve the JR degree of larger than $\bar{n}$ by the winner committee W with $|W| = k = \bar{m} + 1$. If $d \notin W$, no voter in D^+ can be covered since they only approve d. Hence, for the cohesive group $D \cup D^+$, at most $\bar{n}$ voters can be covered, which leads to a contradiction. Thus, $d \in W$. Let $W' = W \setminus \{d\}$, and we have $|W'| = \bar{m}$. If voters in T_j are not covered for some $j \in [\bar{m}]$, at most $\bar{n}$ voters can be covered in the cohesive group $D \cup T_j$, leading to a contradiction. Thus, all voters in group T_j must be covered for each $j \in [\bar{m}]$, indicating that exactly one of the 3 candidates $\{c_{2j-1}, c_{2j}, t_j\}$ for each $j \in [\bar{m}]$ is selected (at least one candidate in each three-candidates group must be selected, we have $\bar{m}$ groups, and we have $|W'| = \bar{m}$). Next, we will show that voters in at least one group among $S_1, \ldots, S_{\bar{n}}$ cannot be covered. Suppose this is not the case. For every $t_j \in W'$, we can use either c_{2j-1} or c_{2j} to replace t_j since t_j only covers group T_j and each of c_{2j-1} and c_{2j} covers at least T_j. Hence, we can find one candidate in $\{c_{2i-1}, c_{2i}\}$ for all $i \in [\bar{m}]$ to cover all groups, implying that there exists a value assignment to the variables (x_i is assigned true if c_{2i-1} is selected, or false otherwise) to make ϕ true, leading to a contradiction. Therefore, at least one group among $S_1, \ldots, S_{\bar{n}}$ cannot be covered. Without loss of generality, we assume that it is S_i. Then, for the cohesive group $S_i \cup D$, at most $\bar{n}$ voters can be covered, leading to a contradiction.

From the proof of NP-hardness, we have that MDJR cannot be approximated in polynomial time to within a factor of $\frac{\bar{n}}{\bar{n}+\bar{m}}$, where $\bar{n}$ is the number of the clauses and $\bar{m}$ is the number of the variables.

To achieve the inapproximation ratios of $(1/k)^{1-\epsilon}$ and $(k/n)^{1-\epsilon}$, we need to make the ratio $\frac{\bar{n}}{\bar{n}+\bar{m}}$ closer to 0. This can be done by modifying the sparse-SAT instance by adding a new clause that consists of many new variables. By increasing the number of new variables in this extra clause, we can make $\bar{m} \gg \bar{n}$. Given that the number of voters is $\bar{m}^2 + O(\bar{n}\bar{m})$ and $k = \bar{m} + 1$, all the theorems are implied.

Given a sparse-SAT instance ϕ with $\bar{n}$ clauses and $\bar{m}$ different variables, we construct another sparse-SAT instance ϕ' with $\bar{n}' = \bar{n}+1$ clauses and $\bar{m}' = \bar{m}+(\bar{n}+\bar{m}+1)^{\lceil \frac{1}{\epsilon} \rceil}$ different variables such that a new clause is added with $(\bar{n}+\bar{m}+1)^{\lceil \frac{1}{\epsilon} \rceil}$ new variables. Obviously, ϕ and ϕ' have the same satisfiability.

Now, we use ϕ' instead of ϕ to construct the ABC voting instance. By our previous analysis, MDJR cannot be approximated in polynomial time to within a factor of $\frac{\bar{n}'}{\bar{n}'+\bar{m}'}$ where $\bar{n}' = \bar{n} + 1, \bar{m}' = \bar{m} + (\bar{n} + \bar{m} + 1)^{\lceil \frac{1}{\epsilon} \rceil}$. Recall that the number of voters and the committee size are $n_{\text{voting}} = \bar{n}'\bar{m}' + \bar{m}'^2 + \bar{n}' + \bar{m}'$ and $k_{\text{voting}} = \bar{m}' + 1$, respectively, which can be reformulated as $\frac{k_{\text{voting}}}{n_{\text{voting}}} = \frac{1}{\bar{n}'+\bar{m}'}$. Since

$$(\bar{n}' + \bar{m}')^{\epsilon} = \left(\bar{n}' + \bar{m} + (\bar{n}' + \bar{m})^{\lceil \frac{1}{\epsilon} \rceil}\right)^{\epsilon} > \bar{n}' + \bar{m} > \bar{n}',$$

MDJR cannot be approximated in polynomial time to within a factor of

$$\frac{\bar{n}'}{\bar{n}' + \bar{m}'} \leq \frac{(\bar{n}' + \bar{m}')^{\epsilon}}{\bar{n}' + \bar{m}'} = \left(\frac{1}{\bar{n}' + \bar{m}'}\right)^{1-\epsilon} = \left(\frac{k_{\text{voting}}}{n_{\text{voting}}}\right)^{1-\epsilon},$$

where $\epsilon > 0$. This proves Theorem 3.

With only minor modifications to the last step above, we can also prove Theorem 4:

$$\left(\frac{1}{\bar{n}' + \bar{m}'}\right)^{1-\epsilon} < \left(\frac{1}{1 + \bar{m}'}\right)^{1-\epsilon} = \left(\frac{1}{k_{\text{voting}}}\right)^{1-\epsilon}. \qquad \square$$

3 PARAMETERIZED COMPLEXITY

The parameterized approach is often used to address problems that are hard to solve in their general form but become more tractable or have improved algorithms when considering specific parameter values. In most scenarios, the committee size k is much smaller than the number of voters. Hence, would it be helpful if we fixed the parameter k?

3.1 W[2]-Hardness with Parameter k.

We show that both MDJR and MDEJR are intractable when the committee size k is specified as a parameter. The proofs for the two theorems below use different techniques, and are available in the full version of our paper.

Theorem 5. MDJR *is W[2]-hard parameterized by k.*

Theorem 6. MDEJR *is W[2]-hard parameterized by k.*

3.2 Fixed-Parameter-Tractability with Parameters k and $c_{\max}$

We have seen that both MDJR and MDEJR are still computationally hard even parameterized by k. Thus, to make the problems tractable, different choices of the parameters or additional parameters are needed.

If we choose the number of candidates m as the parameter, it is easy to verify that both MDJR and MDEJR are fixed-parameter-tractable. To see this, we can enumerate all the $\binom{m}{k}$ committees. For each committee, we can compute the (E)JR degree in $O(2^m m^2 n)$ time [3]. At last, we select the committee that achieves the maximum (E)JR degree.

Another natural choice for the parameter is the maximum achievable (E)JR degree. Fortunately, both MDJR and MDEJR become tractable if parameterized by both k and the maximum (E)JR degree. In the next two sections, we use $c_{\max}$ to denote the maximum (E)JR degree.

3.2.1 Algorithm for MDJR. Our starting point is the algorithm GreedyAV (Algorithm 1) proposed by Aziz et al. [2]. We show the following property for GreedyAV which is the key for our algorithm. It states that the algorithm GreedyAV also gives us the optimal JR degree if n is large enough.

Proposition 4. *Given any instance with the maximum achievable JR degree of $c_{\max}$ and $n > k^2(c_{\max} - 1)$, GreedyAV will output a committee achieving JR degree $c_{\max}$.*

Proof. $n > k^2(c_{\max} - 1)$ implies $c_{\max} < \frac{n}{k^2} + 1$. This proposition then follows from Proposition 2. $\square$

Given any instance with $n > k^2(c_{\max} - 1)$, GreedyAV can achieve the maximum JR degree. Hence, the remaining case is $n \leq k^2(c_{\max} - 1)$. Given any instance $(N, C, \mathbf{A}, k)$, each candidate $c_i \in C$ can be seen as a subset of N including all voters that approve c_i. Hence, there are at most 2^n different types of candidates, implying that every instance corresponds to an equivalent instance with $m \leq 2^n$. Therefore, we can decide whether there exists a committee providing JR degree of c by enumerating all the committees when $n \leq k^2(c-1)$, which can be computed with running time $\binom{m}{k} \leq \binom{2^n}{k} \leq \binom{2^{k^2(c-1)}}{k} = f(k, c)$.

Our algorithm is described in Algorithm 3. We prove in the full version of our paper that Algorithm 3 outputs the committee that achieves the maximum JR degree and runs in time $f(k, c_{\max}) \cdot \text{poly}(m, n)$.

3.2.2 Algorithm for MDEJR. For MDEJR, we prove the following observation in a similar spirit to Proposition 4, which shows that the local search variant of PAV (Algorithm 2) can achieve maximum EJR degree if n is sufficiently large and λ is sufficiently small.

Algorithm 3: MDJR Voting Rule

Input: An instance $\mathcal{I} = (N, C, \mathbf{A}, k))$
Output: A winning committee W of size k

$W \leftarrow$ GreedyAV($\mathcal{I}$)
for $c : \lceil \frac{n}{k^2} \rceil$ *to* $\lfloor \frac{n}{k} \rfloor$ **do**
  Enumerate all those $\binom{m}{k}$ possible committees to see if there is a committee W^* achieving JR degree c;
  if W^* *exists* **then**
    $W \leftarrow W^*$
  else
    return W

Algorithm 4: MDEJR Voting Rule

Input: An instance $\mathcal{I} = (N, C, \mathbf{A}, k)$
Output: A winning committee W of size k

$W \leftarrow \frac{n}{k(k+1)}$-LS-PAV($\mathcal{I}$)
for $c : \lceil \frac{n}{k(k+1)^2} \rceil$ *to* $\lfloor \frac{n}{k} \rfloor$ **do**
  Enumerate all those $\binom{m}{k}$ possible committees to see if there is a committee W^* achieving EJR degree c;
  if W^* *exists* **then**
    $W \leftarrow W^*$
  else
    return W

Proposition 5. *Given any instance with the maximum achievable EJR degree of $c_{\max}$. For any $\lambda \in [0, \frac{n}{k^2})$ satisfying $n > k(k+1)(c_{\max} - 1) + \lambda k^2$, λ-LS-PAV will output the committee achieving EJR degree $c_{\max}$.*

Proof. $n > k(k+1)(c_{\max} - 1) + \lambda k^2$ implies $c_{\max} < \frac{n}{k(k+1)} - \lambda \frac{k}{k+1} + 1$. Proposition 3 implies this proposition. □

Since Proposition 5 holds for every initial committee W in λ-LS-PAV, by considering the initial committee W being the one with the maximum PAV-score and $\lambda = 0$ (in which case the while-loop is never executed), we have the following corollary which may be of independent interest. It shows that, for large enough n, PAV finds a winner committee with the maximum EJR degree. However, the corollary does not hold for all n (as we have remarked before Proposition 3): a counterexample is given in the full version of our paper.

Corollary 1. *Given any instance with the maximum achievable EJR degree of $c_{\max}$. If $n > k(k+1)(c_{\max} - 1)$, PAV has an EJR degree of $c_{\max}$.*

By setting $\lambda = \frac{n}{k(k+1)}$ in Proposition 5, we have the following corollary, which is crucial for our algorithm.

Corollary 2. *Given any instance $(N, C, \mathbf{A}, k)$ with maximum achievable EJR degree $c_{\max}$. If $n > k(k+1)^2(c_{\max} - 1)$, $\frac{n}{k(k+1)}$-LS-PAV outputs a committee with EJR degree $c_{\max}$.*

Based on Corollary 2, we know that Algorithm 2 finds the optimal EJR degree for large enough n, and we can use brute-force search for small n. Thus, we can use a similar way to MDJR to design our algorithm.

In particular, we can decide whether there exists a committee providing EJR degree c by enumerating all the committees when $n \le k(k+1)^2(c_{\max} - 1)$, which can be computed in running time $f(k, c)$. Our algorithm is presented in Algorithm 4, which achieves the maximum EJR degree and runs in time $f(k, c_{\max}) \cdot \text{poly}(m, n)$. The correctness of our algorithm follows from arguments similar to MDJR. The time complexity analysis is also similar, except that we need to show $\frac{n}{k(k+1)}$-LS-PAV runs in polynomial time. To see this, each swap operation can be executed in polynomial time with a minimum score increment by $\lambda = \frac{n}{k(k+1)}$, and the maximum PAV-score is $n \cdot (1 + \frac{1}{2} + \cdots + \frac{1}{k}) = O(n \ln k)$. Thus, the number of while-loop executions is bounded by $O(k^2 \ln k)$.

4 CONCLUSION AND FUTURE WORK

We initialize the study of the (E)JR degree and study its computational complexity and approximability. The (E)JR degree describes (E)JR from a quantitative perspective, which can help us better compare different committees. When explaining (E)JR from a stability perspective, i.e., for any ℓ-cohesive group, if this group deviates and constructs ℓ winners, then at least one member does not want to deviate, as the current satisfaction is already ℓ, which is the maximum satisfaction possible with ℓ winners. However, in reality, if only one person is represented, they can easily be persuaded to deviate by the rest of the group. Hence, if a committee provides a larger (E)JR degree, the possibility of deviation will be reduced. Moreover, we give complete pictures of both optimization problems, from a general NP-hardness with almost tight inapproximability to a parameterized complexity analysis with some natural parameters.

Many potential further works can be explored. For example, one can explore whether the negative results can be circumvented by considering restricted domains of preferences [13, 36]. Another direction is to consider some other quantitative measurements with respect to (E)JR, e.g., using the ratio instead of the number. As we have discussed earlier, our definition of EJR degree using numbers is more aligned with the original definition of EJR, whereas the definition using ratios/fractions, by prioritizing more on ℓ-cohesive groups with large ℓ, is more aligned with the notion of Individual Representation [8] discussed in Sect. 1.3. We believe both the "number version" and the "ratio version" are worth studying. Which choice is better depends on the specific applications.

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