Weighted Envy-free Allocation with Subsidy

Extended Abstract

Haris Aziz
UNSW
Sydney, Australia
haris.aziz@unsw.edu.au

Xin Huang
Kyushu University
Fukuoka, Japan
kidxshine@gmail.com

Kei Kimura
Kyushu University
Fukuoka, Japan
kkimura@inf.kyushu-u.ac.jp

Indrajit Saha
Kyushu University
Fukuoka, Japan
indrajit@inf.kyushu-u.ac.jp

Zhaohong Sun
Kyushu University
Fukuoka, Japan
sunzhaohong1991@gmail.com

Mashbat Suzuki
UNSW
Sydney, Australia
mashbat.suzuki@unsw.edu.au

Makoto Yokoo
Kyushu University
Fukuoka, Japan
yokoo@inf.kyushu-u.ac.jp

ABSTRACT

We consider the problem of fair allocation of indivisible items with subsidies when agents have weighted entitlements. Specifically, we extend the envy-freeability studied in the unweighted case to the weighted envy-freeability and delve deeper into its properties. We first highlight various important differences from the unweighted case, e.g., the sufficient conditions that lead to envy-freeability in the unweighted case do not lead to weighted envy-freeability in the weighted case. We then present various results concerning weighted envy-freeability including general characterizations, algorithms for achieving and testing weighted envy-freeability, lower and upper bounds of the amount of subsidies for weighted envy-freeable allocations. Additionally, we design algorithms that ensure weighted envy-freeability while incorporating other fairness properties, such as weighted envy-freeness up to one item transfer.

KEYWORDS

Fair Division; Envy-free; Subsidy; Weighted Entitlements; Asymmetric Agents

ACM Reference Format:
Haris Aziz, Xin Huang, Kei Kimura, Indrajit Saha, Zhaohong Sun, Mashbat Suzuki, and Makoto Yokoo. 2025. Weighted Envy-free Allocation with Subsidy: Extended Abstract. In *Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Detroit, Michigan, USA, May 19 – 23, 2025*, IFAAMAS, 3 pages.

1 INTRODUCTION

A fundamental problem that frequently arises in various settings is the fair allocation of resources. We consider scenarios where agents have valuations over bundles of indivisible items. The goal is to compute allocations of items that are fair. The gold standard for fairness is envy-freeness, which requires that no agent prefers another agent's allocation. For indivisible item allocation, an envy-free outcome may not exist. There are several approaches to achieve envy-freeness. These include randomisation and the use of monetary subsidies. In this paper, we focus on achieving envy-freeness with monetary subsidies when agents have weighted entitlements.

The literature on envy-free allocation with monetary subsidies / payments / transfers has a long tradition in mathematical economics. For example, the literature on envy-free room-rent division can be viewed as a special case where each agent is supposed to get one item (see, e.g., [8]). More recently, [7] studied the problem of finding allocations for which a minimal amount of subsidies will result in envy-freeness. We revisit envy-freeness with subsidies, with one important extension that agents have weighted entitlements. Weighted entitlements, along with weighted envy-freeness, have been considered in many different contexts in fair division [1, 3–6].

We show that the results under weighted entitlements pose considerable challenges and can often have sharply contrasting results from the unweighted case, i.e., the case of equal entitlements. On the other hand, we also present several results where we generalize some of the celebrated results on envy-freeness with subsidies.

2 MODEL

We consider the setting in which there is a set N of n agents and a set M of m items. We assume each agent $i \in N$ is associated with its weight w_i, where $\sum_i w_i = 1$ and $\forall i \in N, w_i > 0$ hold. Let $w_{min} = \min_i w_i$ and $w_{max} = \max_i w_i$.

Each agent $i \in N$ has a valuation function $v_i : 2^M \longrightarrow \mathbb{R}_0^+$. The function v_i specifies a value $v_i(A)$ for a given bundle $A \subseteq M$. When $A = \{g\}$, i.e., A contains just one item, we often write $v_i(g)$ instead of $v_i(\{g\})$. We assume the valuation functions are *monotone*, i.e., for each $i \in N$ and $A \subseteq B \subseteq M$, $v_i(A) \le v_i(B)$. When we examine the subsidy bounds, we assume the valuation of each agent is bounded, i.e., for any $i \in N$, $A \subseteq M$, $v_i(A) \le |A|$ holds.

The valuation function of an agent i is *super-modular* if for each $i \in N$, and $A, B \subseteq M$, $v_i(A \cup B) \ge v_i(A) + v_i(B) - v_i(A \cap B)$. The valuation function of an agent i is *additive* if for each $i \in N$, and

Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Y. Vorobeychik, S. Das, A. Nowé (eds.), May 19 – 23, 2025, Detroit, Michigan, USA.

$A, B \subseteq M$ such that $A \cap B = \emptyset$, the following holds: $v_i(A \cup B) = v_i(A) + v_i(B)$. The valuation function of agent i is *binary additive* if it is additive and for each $i \in N$ and $g \in M$, $v(g) \in \{0, 1\}$ holds. The valuation function of agent i is *matroidal* if it is a rank function of a matroid [9].

An *allocation* $X = (X_1, \ldots, X_n)$ is a partitioning of the items into n bundles where X_i is the bundle allocated to agent i. We assume allocation X must be *complete*, i.e., $\bigcup_{i \in N} X_i = M$ holds; each item must be allocated to some agent. For an allocation X, the classical weighted social welfare $SW(X)$ is $\sum_{i \in N} w_i \cdot v_i(X_i)$.

An *outcome* is a pair consisting of the allocation and the subsidies received by the agents. Formally, an outcome is a pair (X, p) where $X = (X_1, \ldots, X_n)$ is the allocation that specifies bundle $X_i \subseteq M$ for agent i and $p \in (\mathbb{R}_0^+)^n$ specifies the subsidy p_i received by agent i.

An agent i's *utility* for a bundle-subsidy pair (X_j, p_j) is $v_i(X_j) + p_j$. In other words, we assume quasi-linear utilities. An outcome (X, p) is *envy-free* if for all $i, j \in N$, it holds that $v_i(X_i) + p_i \geq v_i(X_j) + p_j$. An allocation X is *envy-freeable* if there exists a subsidy vector p such that (X, p) is envy-free.

Definition 2.1 (Weighted envy-freeability). An outcome (X, p) is *weighted envy-free* if for all $i, j \in N$:

$$\frac{1}{w_i}(v_i(X_i) + p_i) \geq \frac{1}{w_j}(v_i(X_j) + p_j).$$

An allocation X is *weighted envy-freeable* if there exists a subsidy vector p such that (X, p) is weighted envy-free.

Example 2.2. Assume a family tries to divide inheritance. Agent 1 is the spouse, whose weight is $1/2$. Agents 2 and 3 are children, whose weights are $1/4$. There are two items: g_1 is a house, and g_2 is a car. Some money is also left, but the testament says the money can be divided among agents only to make the outcome weighted envy-free; the remaining amount should be donated to charity. Assume $v_1(g_1) = 100, v_2(g_1) = 70, v_3(g_1) = 0$, and $v_1(g_2) = 40, v_2(g_2) = 60, v_3(g_2) = 0$.

Intuitively, between two agents i and j, w_i/w_j represents the relative importance of agent i against j. Here, the spouse is twice more important than a child, and should get twice more inheritance. Here, agent 3 is not interested in these items, but still cares about the payments. There are two weighted envy-freeable allocations: $(\{g_1, g_2\}, \emptyset, \emptyset)$, i.e., allocating both items to agent 1, and $(\{g_1\}, \{g_2\}, \emptyset)$, i.e., agent 1 obtains g_1, while agent 2 obtains g_2. For the first allocation, we need to pay 65 to agents 2 and 3. For the second allocation, no subsidy is needed; the allocation is weighted envy-free.

Let us introduce a property related to agents' welfare.

Definition 2.3 (Weighted welfare maximizing allocation). We say allocation X maximizes the weighted social welfare if for any allocation X', $SW(X) \geq SW(X')$ holds.

The following example shows that a weighted welfare maximizing allocation may not be weighted envy-freeable.

Example 2.4. Consider the case with two agents 1, 2, with weights $3/4, 1/4$, respectively. There are two identical items. Agent 1 values one item as 90, while agent 2 values one item as 30. The marginal utility for the second item is 0 (these items are substitutes). The weighted social welfare is maximized by allocating one item for each agent. This allocation is not weighted envy-freeable: the subsidy vector p must satisfy $(4/3)(90+p_1) \geq 4(90+p_2)$ ($\Leftrightarrow p_1 \geq 180+3p_2$) and $4(30 + p_2) \geq (4/3)(30 + p_1)$ ($\Leftrightarrow 60 + 3p_2 \geq p_1$), leading to a contradiction that $60 \geq 180$. Moreover, if we consider a partition where each bundle contains only one item, no allocation based on this partition can achieve weighted envy-freeable.

Table 1: We derive upper and lower bounds on worst-case subsidy for each agent in weighted envy-freeable allocations under several valuations.

Valuation	Lower Bound	Upper Bound
General / Super-modular	$m \frac{w_{\max}}{w_{\min}}$	$m \frac{w_{\max}}{w_{\min}}$
Matroidal	$\max\left\{\frac{m}{2}\left(\frac{w_{\max}}{w_{\min}} - 1\right), \frac{w_{\max}}{w_{\min}}\right\}$	$m \frac{w_{\max}}{w_{\min}}$
Additive	$(n-1)\frac{w_{\max}}{w_{\min}}$	$m \frac{w_{\max}}{w_{\min}}$
Identical additive	1	1
Binary additive	$\frac{w_{\max}}{w_{\min}}$	$\frac{w_{\max}}{w_{\min}}$
Additive, identical items	$(n-1)\frac{w_{\max}}{w_{\min}}$	$(n-1)\frac{w_{\max}}{w_{\min}} + 1$

3 OUR RESULTS

Our first contribution is to show that several celebrated results concerning envy-free allocation with subsidies, do not extend to the weighted case:

(1) A welfare maximizing allocation is not necessarily weighted envy-freeable (Example 2.4).
(2) Given any partition of items, there does not necessarily exist a way to allocate the bundles in the partition so it is weighted envy-freeable (Example 2.4).
(3) Envy cannot always be eliminated by providing at most one unit of subsidy for each agent, even under additive valuations.
(4) An allocation that is both weighted envy-freeable and envy-free up to one item may not always exist, even under additive valuations.

Nonetheless, we present a generalized characterization of weighted envy-freeness with subsidies by showing its equivalence with two other carefully specified properties. We show that a weighted envy-freeable allocation can be computed and verified in polynomial time. We show further results for the case of super-modular, matroidal, and additive valuations. In particular, we provide upper and lower bounds for worst-case subsidies in weighted envy-freeable allocations under those valuations. The results are summarized in Table 1. We also propose an algorithm that computes weighted envy-free up to one transfer and weighted envy-freeable allocation for two agents. Finally, we propose a method to achieve a relaxed fairness when we only have a limited amount of subsidies.

The full version of the paper is available in [2].

ACKNOWLEDGMENTS

This work was supported by the NSF-CSIRO grant on "Fair Sequential Collective Decision-Making" (RG230833). The authors affiliated with Kyushu University were partially supported by JSPS KAKENHI Grant Number JP21H04979 and JST ERATO Grant Number JPMJER2301, Japan.

REFERENCES

[1] H. Aziz, H. Chan, and B. Li. 2019. Weighted Maxmin Fair Share Allocation of Indivisible Chores. In *Proceedings of the 28h International Joint Conference on Artificial Intelligence (IJCAI)*.

[2] H. Aziz, X. Huang, K. Kimura, I. Saha, Z. Sun, M. Suzuki, and M. Yokoo. 2024. Weighted Envy-free Allocation with Subsidy. *Preprint arXiv:2408.08711* (2024).

[3] H. Aziz, H. Moulin, and F. Sandomirskiy. 2020. A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. *Operations Research Letters* 48, 5 (2020), 573–578.

[4] M. Babaioff, T. Ezra, and U. Feige. 2023. Fair-share allocations for agents with arbitrary entitlements. *Mathematics of Operations Research* (2023).

[5] M. Chakraborty, A. Igarashi, W. Suksompong, and Y. Zick. 2021. Weighted Envy-freeness in Indivisible Item Allocation. *ACM Transactions on Economics and Computation* 9 (2021), 1–39.

[6] A. Farhadi, M. Ghodsi, M. T. Hajiaghayi, S. Lahaie, D M. Pennock, M. Seddighin, S. Seddighin, and H. Yami. 2019. Fair Allocation of Indivisible Goods to Asymmetric Agents. *Journal of Artificial Intelligence Research* 64 (2019), 1–20.

[7] D. Halpern and N. Shah. 2019. Fair Division with Subsidy. In *Proceedings of the 12th International Symposium on Algorithmic Game Theory (SAGT)*. 374–389.

[8] F. Klijn. 2000. An algorithm for envy-free allocations in an economy with indivisible objects and money. *Social Choice and Welfare* 17 (2000), 201–215.

[9] James Oxley. 2011. *Matroid Theory* (2nd ed.). Oxford Univerity Press.